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BEST MEAN APPROXIMATION TO A 2-DIMENSIONAL KERNEL BY TENSOR PRODUCTS[1]

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We are concerned with the problem

$$\min_{u_i, v_i} \int_0^1 \int_0^1 \left| K(x, y) - \sum_{i=1}^{n} u_i(x) v_i(y) \right| dxdy,$$

where $u_i, v_i \in L^1[0, 1]$, n fixed. The solution of the L^2 version of this problem is a classical result of E. Schmidt [3] (see also Courant and Hilbert [1, p. 161]).

For the class of strictly totally positive kernels K, we are able to show that a best choice of functions $u_1, \ldots, u_n, v_1, \ldots, v_n$ is determined by certain sections $K(x, \xi_1), \ldots, K(x, \xi_n), K(\tau_1, y), \ldots, K(\tau_n, y)$ of the kernel K.

DEFINITION. *A real-valued kernel $K(x, y)$, defined and continuous on $[0, 1] \times [0, 1]$, is called strictly totally positive (S.T.P.) if all its Fredholm minors*

$$K\begin{pmatrix} s_1, \ldots, s_m \\ t_1, \ldots, t_m \end{pmatrix} = \det \| K(s_i, t_j) \|_{i,j=1}^{m}$$

are positive for $0 \leqslant s_1 < \cdots < s_m \leqslant 1$, $0 \leqslant t_1 < \cdots < t_m \leqslant 1$, and all $m \geqslant 1$.

For every $s = (s_1, \ldots, s_m)$, $0 = s_0 < s_1 < \cdots < s_m < s_{m+1} = 1$, define the step function

$$h_s(x) = (-1)^i, \quad s_i \leqslant x < s_{i+1}, i = 0, 1, \ldots, m.$$

Furthermore, let $\|f\|_1 = \int_0^1 |f(x)| dx$, and

$$(Kh_s)(x) = \int_0^1 K(x, y) h_s(y) dy, \quad (K^T h_s)(y) = \int_0^1 K(x, y) h_s(x) dx.$$

The following theorem plays a central role in this work.

THEOREM 1. *Let K be a S.T.P. kernel. Given $n \geqslant 1$, there exists $\xi = (\xi_1, \ldots, \xi_n)$, $0 < \xi_1 < \cdots < \xi_n < 1$, such that for any $t = (t_1, \ldots, t_n)$, $0 < t_1 < \cdots < t_n < 1$,*

$$\|Kh_\xi\|_1 \leqslant \|Kh_t\|_1.$$

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Moreover, Kh_ξ *has exactly n distinct sign changes at* $\tau = (\tau_1, \ldots, \tau_n)$, $0 < \tau_1 < \cdots < \tau_n < 1$, *and*

(1) $\operatorname{sgn} Kh_\xi = h_\tau$,

(2) $\operatorname{sgn} K^T h_\tau = h_\xi$.

(When Kh_ξ or $K^T h_\tau$ are zero in (1) or (2), we assign a value to the sgn so that the equations are valid.)

COROLLARY. *Let* $\tau = (\tau_1, \ldots, \tau_n)$ *be the τ-point defined in Theorem* 1. *Then,*

$$\|Kh_\xi\|_1 = \|K^T h_\tau\|_1 \leqslant \|K^T h_s\|_1$$

for every s-point, $s = (s_1, \ldots, s_n)$, $0 < s_1 < \cdots < s_n < 1$.

We are now prepared to state the main theorem. To this end observe that the function

$$E(x, y) = K\begin{pmatrix} x, & \tau_1, & \ldots, & \tau_n \\ y, & \xi_1, & \ldots, & \xi_n \end{pmatrix} \Big/ K\begin{pmatrix} \tau_1, & \ldots, & \tau_n \\ \xi_1, & \ldots, & \xi_n \end{pmatrix} \tag{3}$$

(where ξ and τ are obtained from Theorem 1) may be expressed as

$$= K(x, y) - \sum_{i,j=1}^{n} c_{ij} K(x, \xi_i) K(\tau_j, y),$$

where

$$c_{ij} = (-1)^{i+j} K\begin{pmatrix} \tau_1, & \ldots, & \tau_{j-1}, & \tau_{j+1}, & \ldots, & \tau_n \\ \xi_1, & \ldots, & \xi_{i-1}, & \xi_{i+1}, & \ldots, & \xi_n \end{pmatrix} \Big/ K\begin{pmatrix} \tau_1, & \ldots, & \tau_n \\ \xi_1, & \ldots, & \xi_n \end{pmatrix}.$$

Therefore

$$\begin{aligned} E_{1,1}(K) &\equiv \min_{u_i, v_i} \int_0^1 \int_0^1 \left| K(x, y) - \sum_{i=1}^{n} u_i(x) v_i(y) \right| dx\,dy \\ &\leqslant \int_0^1 \int_0^1 |E(x, y)|\,dx\,dy. \end{aligned} \tag{4}$$

Actually, we have

THEOREM 2.

$$\begin{aligned} E_{1,1}(K) &= \int_0^1 \int_0^1 |E(x, y)|\,dx\,dy = \|Kh_\xi\|_1 \\ &= \int_0^1 \int_0^1 \left|K(x, y) - \sum_{i=1}^{n} u_i^0(x) v_i^0(y)\right| dx\,dy, \end{aligned}$$

where $u_i^0(x) = K(x, \xi_i)$, *and* $v_i^0(y) = \sum_{j=1}^{n} c_{ij} K(\tau_j, y)$.

PROOF. By the Hobby-Rice Theorem [2], we know that given any n functions $v_1, \ldots, v_n \in L^1[0, 1]$, there exists a $t = (t_1, \ldots, t_k)$, $0 \leqslant k \leqslant n$, such that $\int_0^1 v_i(y)h_t(y)\,dy = 0$, $i = 1, \ldots, n$. Let $h(x, y) = h_t(y)\operatorname{sgn}(Kh_t)(x)$. Then, for any $u_1, \ldots, u_n \in L^1[0, 1]$,

$$\|Kh_\xi\|_1 \leqslant \int_0^1 |(Kh_t)(x)|\,dx = \int_0^1\int_0^1 \left(K(x, y) - \sum_{i=1}^n u_i(x)v_i(y)\right) h(x, y)\,dx\,dy$$

$$\leqslant \int_0^1\int_0^1 \left|K(x, y) - \sum_{i=1}^n u_i(x)v_i(y)\right| dx\,dy.$$

Since $u_1, \ldots, u_n, v_1, \ldots, v_n$ were arbitrarily chosen in $L^1[0, 1]$, we have $\|Kh_\xi\|_1 \leqslant E_{1,1}(K)$. Also, in view of (1), (2), and (3),

$$\int_0^1\int_0^1 |E(x, y)|\,dx\,dy = \int_0^1\int_0^1 E(x, y)h_\tau(x)h_\xi(y)\,dx\,dy$$

$$= \int_0^1 (Kh_\xi)(x)h_\tau(x)\,dx - \sum_{i,j=1}^n c_{ij}(K^T h_\tau)(\xi_i)(Kh_\xi)(\tau_j)$$

$$= \int_0^1 |(Kh_\xi)(x)|\,dx = \|Kh_\xi\|_1,$$

which, together with (4), finishes the proof.

Full details, extensions, and the relationship of this problem to n-widths will appear elsewhere.

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