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RESEARCH ANNOUNCEMENTS

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[*Note.* The criteria described above are new; in the transition period they will not necessarily be met by the announcements appearing in this issue.]

ON ALMOST MINIMALLY ELLIPTIC SINGULARITIES

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Let p be the unique singularity of a normal two-dimensional Stein space V. Let $\pi: M \longrightarrow V$ be a resolution. It is known that $\dim H^1(M, \mathcal{O})$ is independent of resolution. In [1], M. Artin developed a theory for those singularities with $\dim H^1(M, \mathcal{O}) = 0$. Recently Laufer [5] has developed a theory for those singularities which has $\dim H^1(M, \mathcal{O}) = 1$ and ${}_V\mathcal{O}_p$ (the local ring of germs of holomorphic functions at p) is Gorenstein. Although the title of this paper is *Almost minimally elliptic singularities*, our main interest is to build up a theory for those singularities which has $\dim H^1(M, \mathcal{O}) = 2$ and ${}_V\mathcal{O}_p$ is Gorenstein. All undefined terms and notations are standard in [5] and [7]. It is a pleasure to acknowledge the help and encouragement of Professor Henry Laufer in this research. We also like to thank Professor Bennett, Professor Kuga and Professor Sah for their discussion of Mathematics.

1. General theory. Throughout this paper, E will denote the minimally elliptic cycle and Z will denote the fundamental cycle. A will denote $\pi^{-1}(p)$.

DEFINITION 1.1. Let $\pi: M \longrightarrow V$ be the minimal good resolution of a normal two-dimensional Stein space with p as its only weakly elliptic singular point. Suppose p is not a minimally elliptic singularity, i.e. $|E| \neq \pi^{-1}(p) = \bigcup A_i$. If for all $A_i \not\subseteq |E|$ and $A_i \cap |E| \neq \varnothing$, then $A_i \cdot Z < 0$. We call p an *almost minimally elliptic singularity.*

PROPOSITION 1.2. *Suppose p is an almost minimally elliptic singularity and ${}_V\mathcal{O}_p$ is Gorenstein, then* $H^1(M, \mathcal{O}) = \mathbf{C}^2$.

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In [7], we know that $H^1(M, \mathcal{O}) = \mathbf{C}^2$ and ${}_V\mathcal{O}_p$ Gorenstein imply that p is a weakly elliptic singularity. But it is not true that $H^1(M, \mathcal{O}) = \mathbf{C}^2$ and ${}_V\mathcal{O}_p$ Gorenstein will imply that p is an almost minimally elliptic singularity. However the following is true.

THEOREM 1.3. *Let π: $M \longrightarrow V$ be the minimal good resolution of normal two-dimensional Stein space with p as its only weakly elliptic singularity. Suppose $H^1(M, \mathcal{O}) = \mathbf{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Then p is an almost minimally elliptic singularity if and only if $H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) = \mathbf{C}$.*

DEFINITION 1.4. Let $D = \Sigma d_iA_i$ be a positive cycle. Let $B \subseteq |D|$. Then $D/B = \Sigma f_iA_i$ is a positive cycle where $f_i = d_i$ if $A_i \subseteq B$ and $f_i = 0$ if $A_i \nsubseteq B$.

THEOREM 1.5. *Use the notation of Theorem 1.3. Suppose $H^1(M, \mathcal{O}) = \mathbf{C}^2$, $H^1(|E|, \mathbf{Z}) = 0$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0}, Z_{B_1}, \ldots, Z_{B_l}, Z_E$ be the elliptic sequence (cf. [7]). Let D be the subvariety of B_l consisting of those irreducible components $A_i \subseteq B_l$ such that $A_i \cap |E| \neq \varnothing$. If $Z/D = Z_{B_l}/D$, then $l = 0$, i.e. p is an almost minimally elliptic singularity.*

2. Calculation of multiplicities and Hilbert functions. Suppose $H^1(M, \mathcal{O}) = \mathbf{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. We identify the maximal ideal and in particular, we get a formula for the multiplicity of the singularity. We also calculate the Hilbert function of ${}_V\mathcal{O}_p$. In particular, the dimension of the Zariski tangent space is computed. Hence we know the lowest possible embedding dimension of the singularity. We remark that all the results are sharp.

THEOREM 2.1. *Use the notation of Theorem 1.3. Suppose $H^1(M, \mathcal{O}) = \mathbf{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0} = Z, Z_{B_1}, \ldots, Z_{B_l}, Z_E$ be the elliptic sequence. Then $m\mathcal{O} \subseteq \mathcal{O}(-\Sigma_{i=0}^{l} Z_{B_i})$. If $Z_E \cdot Z_E \leqslant -2$, then $m\mathcal{O} = \mathcal{O}(-\Sigma_{i=0}^{l} Z_{B_i})$. If $Z_E \cdot Z_E \leqslant -3$, then $m^n \simeq H^0(A, \mathcal{O}(-n(\Sigma_{i=0}^{l} Z_{B_i})))$, $n \geqslant 0$ and $\dim m^n/m^{n+1} = -n \cdot (\Sigma_{i=0}^{l} Z_{B_i}^2)$, $n \geqslant 1$.*

As a consequence of Theorem 1.5 and Theorem 2.1, we prove the partial converse of Proposition 1.2.

THEOREM 2.2. *Suppose p is a hypersurface singularity. If $H^1(M, \mathcal{O}) = \mathbf{C}^2$ and $H^1(A, \mathbf{Z}) = 0$, then p is an almost minimally elliptic singularity.*

3. Absolutely isolatedness of almost minimally elliptic singularities. In this paper we shall say that a two-dimensional isolated singularity is absolutely isolated if it can be resolved by means of a sequence of σ processes with centers at points. We have the following absolutely isolatedness theorem.

THEOREM 3.1. *Let π: $M \longrightarrow V$ be the minimal good resolution of normal two-dimensional Stein space with p as its only almost minimally elliptic singularity. If $Z_E \cdot Z_E \leqslant -3$ and ${}_V\mathcal{O}_p$ is Gorenstein, then p is absolutely isolated. Moreover, blow-up p at its maximal ideal yields exactly those curves A_i such that $A_i \cdot Z < 0$. The singularities remaining after the blow-up are the rational double* *points and a minimally elliptic singularity corresponding to delecting the A_i with $A_i \cdot Z < 0$ from the exceptional set. The self intersection number of the fundamental cycle of the minimally elliptic singularity is less than or equal to* -3.

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