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Classification theory of algebraic varieties and compact complex spaces, by Kenji Ueno, Lecture Notes in Math., no. 439, Springer-Verlag, Berlin, Heidelberg, New York, 1975, 278 + xix pp., $12.10.

Looking for a research area in which you can start at the ground floor? This is it–the classification of varieties in dimensions three and higher. Of course the prerequisites might sound a little stiff: algebraic geometry, complex analytic geometry, and the "classical" classification theory in dimensions one and two, but in reality it's not as bad as one might fear. The author of the text under review gave an informal course on the subject at the University of Mannheim in 1972, and the lecture notes (by P. Cherenack) form the basis of the text. At the very least this book is a way to take a peek at what is going on in this new field, and maybe even it is a way to get into it.

The first pleasant surprise the neophyte encounters in the study of geometry is that the two main categories in which geometers work–the analytic category and the algebraic category–actually have a very large overlap. Thus studying one category allows one to absorb "by osmosis" results in the other. The terminology is sometimes different; for example, here is a short, *rough*, transliteration guide:

Algebraist's term	Analysts' term
complete	compact
separated	Hausdorff
nonsingular variety	manifold
algebraic space (complete)	Moishezon space (compact)
projective variety	Kähler variety (compact)
rational map	meromorphic map
birational map	bimeromorphic map
curve	Riemann surface

Many other terms (e.g. proper, normal, irreducible) have the same geometric content but a particular formulation of a definition in one category may not make sense in the other. For example the characterization of a proper map as being universally closed works in both categories, while the characterization that the inverse image of compact is compact is too weak in algebraic geometry. The author basically works in the analytic category with compact, reduced and irreducible, complex spaces, but he devotes much of the first chapter stating the details of the correspondences which connect these two categories. He also reviews the basic theorems in geometry–Stein factorization, Grauert's theorem on the coherence of the higher direct images of a coherent sheaf, resolution of singularities, modifications, etc. There are of course few proofs given here, but references to the literature are given, and the presence of the statements greatly enhances the clarity of the entire exposition.

The adjective "algebraic" is used when dealing with compact Moishezon spaces. These are complex analytic spaces V where the **C**-transcendence degree $a(V)$ of $\mathbf{C}(V)$, the field of meromorphic functions $V \to \mathbf{C}$, equals the dimension $\dim V$ of the space. (A reducible space is Moishezon if each irreducible component is Moishezon.) Any projective variety is Moishezon. If M is any complex space, one can easily construct a fiber map (the author's term for a surjective morphism with connected fibers) $f\colon M^* \to V$ of nonsingular spaces where M^* is bimeromorphically equivalent to M, V is projective, and f induces an isomorphism $\mathbf{C}(V) \simeq \mathbf{C}(M^*) = \mathbf{C}(M)$. Such a map is called an algebraic reduction and provides a link between nonalgebraic and algebraic spaces. (Cf. [**5**] for a nice application.)

The first unpleasant surprise one discovers in studying classification theory is that it is not a classification at all, but rather an informative grouping of varieties by means of certain birational invariants. This grouping has much the flavor of grouping nonsingular projective curves according to their genus, but as we shall see this grouping in higher dimensions is not nearly as good as grouping curves by their genus!

The first and most basic invariant which is studied in this theory is the Kodaira dimension of a vareity V, thus it is worth understanding its definition. To simplify, let's assume V is nonsingular (in addition to the standing assumptions of compactness, irreducibility, etc.) of dimension n. If L is a line bundle on V then we can define a meromorphic mapping $\phi_L\colon V \to \mathbf{P}^N$ by $\phi_L(z) = (\phi_0(z)\colon \phi_1(z)\colon \cdots \colon \phi_N(z))$ where $\{\phi_0, \phi_1, \ldots, \phi_N\}$ is a basis for $H^0(V, L)$, the group of global sections of L. If $V(L)$ denotes the image of ϕ_L, we define the *L-dimension of V* to be $\kappa(L, V) = \max\{\dim V(L^{\otimes m})\colon m > 0\}$. (The dimension of the empty set is taken to be $-\infty$.) When we take L to be the canonical bundle Ω_V^n of n-forms on V we get the *Kodaira dimension of V*, $\kappa(V) = \kappa(\Omega_V^n, V)$. For example, for a curve C: $\kappa(C) = -\infty$ if C is rational (genus $g = 0$), $\kappa(C) = 0$ if C is elliptic (genus $g = 1$), and $\kappa(C) = 1$ otherwise (genus $g \geqslant 2$). The Kodaira dimension is the basis for the grouping of varieties mentioned already, and here with curves we can see that the Kodaira dimension is a very crude invariant indeed. In the case of surfaces, just how the Kodaira dimension fits into the classification scheme is well understood (cf. [**2**, p. 415] for a nice table): If K denotes the divisor class of the canonical bundle Ω_V^2 on a surface V then $\kappa(V) \geqslant 0$ if and only if $K \cdot D \geqslant 0$ for all divisors $D \geqslant 0$ in which case $0 \leqslant \kappa(V) \leqslant 1$ if $K^2 = 0$ and $\kappa(V) = 2$ if $K^2 > 0$. Another nice example is when V is an n dimensional "complete intersection" defined in $\mathbf{P}^{m+n}$ by homogeneous polynomials $F_1, \ldots, F_m$ of degree $d_1, \ldots, d_m$ then $\kappa(V) = -\infty, 0, n$ according to whether $\sum d_i$ is $\langle\, , =, \rangle$ than $m + n + 1$. Thus three-folds in $\mathbf{P}^4$ of degree $\leqslant 4, 5, \geqslant 6$ have Kodaira dimension $-\infty$, 0, and 3 respectively. Of course, when L is ample, i.e. when some tensor power of L gives a projective embedding, then $\kappa(L, V) = \dim V$. This occurs for instance when $V = D/\Gamma$ where D is a bounded domain in $\mathbf{C}^n$ and Γ is a discontinuous group of automorphisms acting freely on D with compact quotient. In this case Kodaira has shown that the canonical bundle is ample. These and other examples are discussed in detail in Chapter two of the text.

The author spends much of his book discussing both results and conjectures on these dimensions. For example one has the following facts:

1. $\kappa(L, V) \leqslant a(V) \leqslant \dim V$.

2. If $f\colon V \to W$ is a fibre map, V and W nonsingular, then for any line bundle L on V there is a dense open subset U of W such that for $w \in U$

$$\kappa(L, V) \leqslant \kappa(L_w, V_w) + \dim W$$

where L_w denotes the restriction of L to the fiber $V_w = f^{-1}(w)$.

3. If f is as in (2) and M is a line bundle on W then $\kappa(f^* M, V) = \kappa(M, W)$.

4. $\kappa(V_1 \times V_2) = \kappa(V_1) + \kappa(V_2)$.

Despite the plethora of such results, much remains unknown about even the Kodaira dimension. For example the inequality in (2) is rather weak, because the term $\dim W$ does not involve L. In the case of the Kodaira dimension one has the following conjecture for spaces V of dimension n:

C_n: For a fiber map $f\colon V \to W$ between nonsingular algebraic varieties, $\kappa(V) \geqslant \kappa(W) + \kappa(V_w)$ for a dense open set of w in W.

Putting this together with (2) above would give $\kappa(W) \leqslant \kappa(V) - \kappa(V_w) \leqslant \dim W$. This holds for $n = \dim V = 2$, but it is unknown in higher dimensions except in special cases. The author even spends an entire chapter proving this inequality in the case that f is a fiber bundle map whose fiber F is algebraic and whose structure group is the entire automorphism group $\mathrm{Aut}(F)$ of F. While this case may seem rather special, the author concludes this chapter with a counterexample to this inequality where $f\colon V \to E$ is a fiber bundle over an elliptic curve E, the fiber F is a nonalgebraic three dimensional torus, and the structure group is $\mathrm{Aut}(F)$.

The first three chapters conclude with a proof of Iitaka's "fundamental theorem of classification theory". While the name is a little pompous, the theorem is nice. Under the hypothesis $\kappa(V) > 0$, the theorem asserts the existence and essential uniqueness of a modification (= a blowing up) $V^* \to V$ and a surjective map $f\colon V^* \to W^*$ where W^* is nonsingular and projective, $\dim W^* = \kappa(V)$, and for a dense subset U of W^* the fibers of f over U are irreducible, nonsingular, and have Kodaira dimension zero. (There is also a similar theorem for the L-dimension.) Of course the map f is induced by one of the pluri-canonical mappings ϕ_L where $L = \Omega_V^{n \otimes m}$ for some large $m > 0$, and the key point of the theorem is that most fibers will have Kodaira dimension zero and geometric dimension $\dim V - \kappa(V)$. For example if $\dim V = 2$ there are only two cases: $\kappa(V) = 1$ and $\kappa(V) = 2$. In the first case the pluri-canonical mappings (for large m) have curves as images and genus one ($\kappa = 0$) curves ($\dim = 2 - 1 = 1$) for fibers. In the second case when $\kappa(V) = 2$, the canonical bundle is not quite ample, but for large m the pluri-canonical mappings $\phi_L\colon V \to W$ are isomorphisms except along certain curves in V which get collapsed to points in W (these points are in fact rational singularities). Here we see why both V and W are replaced by V^* and W^* in the conclusion of the theorem. Note also that the general fiber here is just a point as is always the case when $\kappa(V) = \dim V$. Thus the moral of the theorem is that one way to get a classification theory in dimensions $n \geqslant 2$ is to proceed as follows: Kodaira dimensions $\kappa = -\infty, 0, n$ have to be handled separately, while the intermediate cases $\kappa = 1, \ldots, n-1$ are reduced to these by studying the fiber map f given by the theorem. Note that when $\kappa = n$, the space is automatically algebraic by property (1) above.

There is a third fiber map used in this theory. Recall that to each compact complex manifold V there is a complex torus $A(V)$ (which is an abelian variety if V is algebraic) and a map $\alpha: V \to A(V)$ with the following universal property. If $g: V \to T$ is any map of V into a complex torus T then there is a unique Lie group homomorphism $h: A(V) \to T$ and a unique $t \in T$ such that $g(x) = h(\alpha(x)) + t$ for all $x \in V$. Thus α is essentially unique and is called "the" albanese mapping; $A(V)$ is called the albanese torus. Unfortunately in general α is not surjective, nor does $V \to \alpha(V)$ have connected fibers. The third fiber map β comes from the Stein factorization of α:

$$\begin{array}{ccc} V & \longrightarrow & \alpha(V) \subset A(V) \\ \downarrow & \nearrow & \\ \beta(V) & & \end{array}$$

One has $\dim A(V) \leqslant \dim H^0(V, \text{closed 1 forms}) \leqslant \dim H^0(V, \Omega^1_V)$ with equality when V is algebraic. To study α and β the author studies subvarieties of complex tori and concludes for example, if V is nonsingular then $\kappa(\beta(V)) \geqslant \kappa(\alpha(V)) \geqslant 0$ and $\kappa(\alpha(V)) = 0$ if and only if α is surjective.

To get a feeling for these results, let's suppose conjecture C_n above is true. Then $\kappa(V) \geqslant \kappa(V_w) + \kappa(\beta(V)) \geqslant \kappa(V_w) + \kappa(\alpha(V))$. Suppose also $\kappa(V) = 0$, then there is a nonzero element $\gamma \in H^0(V, \Omega_V^{n \otimes m})$ for some $m > 0$, and a not too difficult argument shows that almost everywhere γ induces a nonzero element in the corresponding group of the fiber, hence we may assume $\kappa(V_w) \geqslant 0$. Collecting these inequalities together implies that $0 \geqslant \kappa(V_w) + \kappa(\alpha(V)) \geqslant 0$ hence $\kappa(V_w) = 0$ and $\kappa(\alpha(V)) = 0$. By the result mentioned in the previous paragraph we can conclude that α is surjective, and also we have that the fibres (of β) have Kodaira dimension zero. The author's strongest conjecture about algebraic varieties of Kodaira dimension zero concludes not only what we have here using conjecture C_n, but also that the fibres of α are connected, i.e. that $\alpha = \beta$. These conjectures are related to other, more striking, conjectures discussed at the end of Chapter four.

In addition to results already mentioned, the remainder of the book contains rather special results. There are some results on complex spaces whose algebraic dimension, $a(V)$, is zero. These are the spaces which are the least algebraic in terms of having meromorphic functions. Here $\kappa = -\infty$ or 0, the albanese map α is surjective, and the albanese torus also has algebraic dimension zero. Thus one can squeeze out some information on what kinds of fibers α has in this case. There are special results on Kummer manifolds–nonsingular models of quotients of an abelian variety by a finite group of analytic automorphisms. Here $\kappa \leqslant 0$, the albanese is a fiber map, and $\kappa = 0$ if and only if there exists a birationally equivalent manifold whose albanese mapping is an analytic fiber bundle whose fiber is a Kummer manifold with Kodaira dimension zero. In this section also are some calculations with the Kummer manifolds associated to $\mathbf{C}^n$ modulo a cyclic action. There is a section on complex parallelizable manifolds–which are all of the form G/Γ where G is a 1-connected complex Lie group and Γ is a discrete subgroup. There is a section on complex structures on a product of two odd dimensional spheres, e.g. the Hopf manifolds structures on $S^1 \times S^{2n-1}$. Finally, if these results aren't miscellaneous enough, there is a section entitled "*Miscellaneous results*" and a final section on the classification of surfaces which might be useful to glance at but should be studied elsewhere.

So to where has classification theory progressed? For curves there are even moduli spaces–varieties whose points are in natural one-one correspondence with the isomorphism classes of the objects you are classifying–which are "understood" for genus $g \leqslant 10$ (cf. [**6**] for an utterly delightful account). While these moduli spaces still offer good research problems, the classification per se is complete. The rough classification for surfaces, while not completely finished, is certainly a rather sophisticated one. Moduli spaces only exist now for surfaces of general type ($\kappa = \dim = 2$) and are not well understood. None the less, the theory is complete enough to want to look at three-folds. Here, the subject is miserable in its infancy. The five possible Kodaira dimensions give five fertile and essentially unplowed areas of research:

$\kappa = -\infty$. There is no "Castelnuovo's criterion" for a ruled variety (bimeromorphically equivalent to $\mathbf{P}^1 \times V$) as there is for surfaces, nor is there a criterion for rationality (bimeromorphically equivalent to $\mathbf{P}^n$). The flurry of activity on cubic three-folds a couple years ago (cf. [**1**], [**3**], [**4**]) produced some methods to distinguish a rational from a unirational (finite image of a rational) variety. Thus we know that cubic three-folds in $\mathbf{P}^4$–which are easily seen to be unirational–are not rational, but it is still not known whether the quartic three-folds (still $\kappa = -\infty$) are all unirational. Also, many classical claims (cf. [**8**]) about Fano threefolds (where the dual of Ω_V^3 is ample) remain unproven despite recent work (cf. [**10**]).

$\kappa = 0$. As we saw above the albanese mapping comes into play. Just glancing at the classification of surfaces of Kodaira dimension zero gives another reason to expect abelian varieties and tori to enter the picture. Indeed, one of the conjectures in the text is that an algebraic three-fold in this class has a finite (ramified) cover which is birationally equivalent to the product of an abelian surface and an elliptic curve. One technical problem here is that it is not necessarily true for three-folds with $\kappa = 0$ that a multiple of the canonical class is trivial as is the case for surfaces with $\kappa = 0$. Curiously even quintic threefolds seem to be unstudied.

$\kappa = 1$. Here the high pluri-canonical mappings have curves as images and surfaces of Kodaira dimension zero as fibers, but studies of families of surfaces aren't too common (cf. however, [**7**], [**9**]). Actually we have several moduli problems here depending upon where you are in the classification of these surfaces, and none of these moduli problems is well understood.

$\kappa = 2$. In this case the high pluri-canonical mappings have surfaces as images and elliptic curves as fibres. One would think that this case would be amenable to attack. It should be noted that the author has generalized Kodaira's "canonical bundle formula" in this situation when the branch locus has normal crossings to give an explicit formula for a multiple of the canonical class on the three-fold [this is (11.8.1) of the text].

$\kappa = 3$. For surfaces of maximal Kodaira dimension the higher pluricanonical mappings are morphisms; it is unknown whether this is the case for three-folds. It is also not known whether deformations of such three-folds still have $\kappa = 3$. One would hope that a moduli space would exist for such three-folds as one does for surfaces of general type; indeed, some work towards this goal has been accomplished now. Again the case of sextic three-folds in $\mathbf{P}^4$ doesn't seem to have been studied.

The author has indeed provided the mathematical community with a valuable manuscript. It could well serve as the basis for independent study or for a seminar; although, for a seminar topic perhaps a detailed look at the classification theory of surfaces would be more profitable. As a reference it serves best as a guide to the literature; although one notable feature is that it includes some new and better proofs of published results.

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GAYN B. WINTERS

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Introduction to axiomatic quantum field theory, by N. N. Bogolubov, A. A. Logunov, and I. T. Todorov, Mathematics Physics Monograph, no. 18, W. A. Benjamin, Inc., Reading, Massachusetts, 1975, xxvi + 708 pp., $32.50.

Let us begin with a brief history of why physicists attach great importance to the quantum theory of fields. Dirac, Heisenberg and other great scientists conceived this theory as a synthesis of two extremely fruitful ideas. On the one hand, relativistic quantum mechanics (the Dirac equation) had extended Schrödinger mechanics to predict quantitatively the fine structure of the hydrogen atom spectrum. It also suggested the existence of antimatter. On the other hand, classical field theory (Maxwell's equations for electromagnetism and the Newton-Einstein theory of gravity) provided the theoretical basis for macroscopic physics. The hypothesis of quantum field theory was that