BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 6, November 1976

THE INVARIANTS OF $n \times n$ MATRICES

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Communicated by Olga Taussky Todd, May 11, 1976

The purpose of this paper is threefold: to classify the invariants of matrices, study their relations, and understand the quotient varieties that these invariants classify (i.e. to determine when two sets of matrices have the same invariants). We obtain fairly complete answers to these questions. The groups with respect to which we study invariants are the classical algebraic groups $\mathrm{GL}(n, \mathbf{C})$, $O(n, \mathbf{C})$, $\mathrm{Sp}(2n, \mathbf{C})$ and their maximal compact subgroups. The action on m-tuples of matrices is always the diagonal action with conjugation in each component. We list the main results:

(1) The invariants of matrices $X_1, X_2, \ldots, X_i, \ldots$ under $\mathrm{GL}(n, \mathbf{C})$ are generated by the monomials $\mathrm{Tr}(X_{i_1} X_{i_2} \cdots X_{i_s})$. More generally the matrix valued invariants are generated by these monomials and the variables X_i.

(2) Every relation among such invariants is a consequence of the Hamilton-Cayley Theorem.

For the case of $\mathrm{GL}(n, \mathbf{C})$ the study of the quotient variety was carried out by A. Artin, who proved that two m-tuples of matrices $(X_1, \ldots, X_m)$, $(Y_1, \ldots, Y_m)$ have the same invariants if and only if the "semisimple parts" (i.e. associated semisimple representations) are conjugate.

For the unitary group $U(n, \mathbf{C})$ one has:

(3) The invariants are generated by the monomials $\mathrm{Tr}(U_{i_1} U_{i_2} \cdots U_{i_s})$ where $U_i = X_i$ or $U_i = \overline{X}_i^t$.

(4) Every relation is a consequence of the Hamilton-Cayley Theorem.

(5) Two sets of matrices $(X_1, \ldots, X_i, \ldots)$, $(Y_1, \ldots, Y_i, \ldots)$ are conjugate under $U(n, \mathbf{C})$ if and only if they have the same invariants.

(6) The invariants can be given arbitrarily subject only to:
(a) The formal algebraic relations and
(b) The inequalities $\mathrm{Tr}(p\overline{p}^t) \geqslant 0$, p a polynomial in $X_i, \overline{X}_i^t$.

AMS (MOS) subject classifications (1970). Primary 16A64, 14D20, 16A38, 16A16.
Key words and phrases. Invariants of matrices, semisimple representations, polynomial identities, Azumaya algebras.

For the orthogonal group and symplectic group the results are similar; we list them for $O(n, \mathbf{C})$.

(7) The invariants of $X_1, X_2, \ldots, X_i, \ldots$ under $O(m, \mathbf{C})$ are generated by $\mathrm{Tr}(U_{i_1} \cdots U_{i_k} \cdots)$, $U_i = X_i$ or $U_i = X_i^t$, and similar results hold for matrix valued invariants.

(8) The relations among such invariants are consequences of a finite explicit list of independent relations.

(9) $(X_1, \ldots, X_i, \ldots)$, $(Y_1, \ldots, Y_i, \ldots)$ have the same invariants if and only if the semisimple representations associated are $O(m, \mathbf{C})$ equivalent.

(10) Over the real numbers (9) becomes stronger; the tuples have the same invariants if and only if they are conjugate.

We draw some consequences for polynomial identities of matrices.

Finally we analyze the irreducible sets of matrices in terms of their invariants and show that these are stable points; in fact on such points the action is locally (in the étale topology) a product.

The technique of the proof is:

(a) reduction to the multilinear case (this is just Aronhold's rule);

(b) identification of matrices with tensors, i.e. $\mathrm{End}(V) \cong V \otimes V^*$ and reduction of our theorems to the classical first and second fundamental theorems of vectors and forms.

In the second part of the development we have to interpret the notions introduced in terms of module theory and study the semisimple representations associated to a given nonsemisimple representation. The study of the irreducible representations uses the theory of Azumaya algebras, and follows closely the ideas of M. Artin on Azumaya algebras and finite-dimensional representations of rings (J. Algebra 2 (1969), 532–566).

Finally, we should note that item (2) has been obtained independently by Y. Razmyslov.

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