interchange of ideas with physics. Nowadays, physicists are more interested in indefinite Riemannian metrics. Again, the authors' narrow viewpoint has precluded anything on this topic. In fact, in parallel with this work–and with almost no interdisciplinary communication–physicists interested in applications of general relativity to cosmology and astrophysics have used similar techniques to prove singularity theorems about nonpositive Riemannian metrics. See the book, *The large-scale structure of space-time*, by Hawking and Ellis.

This book is a historical landmark in the sense that it is the first to concentrate on the successes of post-World War II differential geometry. Every book published before has been more-or-less an attempt to understand the work of the great masters in the light of the modern sensibility. Above all else, we have had to struggle to understand Elie Cartan! (One can even trace the spirit of this book back to Cartan, particularly *Géométrie des éspaces de Riemann*.) As I have already mentioned, the great successes recounted here have been achieved at the expense of partially, and perhaps only temporarily, abandoning the sweeping outlook of the classical work. One has only to compare this material to that in the collected works of Cartan and Lie and in Darboux' *Théorie des surfaces* to realize how much of our heritage has been dumped overboard. Perhaps this is due to our overemphasis on maintaining our status in the eyes of our big brothers, the topologists. I recall that when I was a student in the fifties everyone almost went around chanting, in Red Guard fashion: global good, local bad. Of course, this fanaticism had the happy consequence of leading to this impressive work; one will never know what might have been achieved if differential geometry had kept its traditional orientation. I would now want to ask how the techniques so precisely and powerfully developed in this modern work can be applied to the broader classical problems. My own guess is that most likely the field awaiting conquest is the geometric theory of nonlinear partial differential equations. (For example, it is not at all well known that Darboux' formidable treatise contains much more about this subject than it does about the theory of surfaces!) Perhaps the seeds to great advances in this field–and the recent discovery of "solitons" suggests that it is also of great interest for physics–lie in Darboux just as the seeds that grew to these magnificent comparison theorems were buried in the work of Jacobi, Riemann and Ricci.

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Discrete-parameter martingales, by Jacques Neveu, North-Holland, Amsterdam; American Elsevier, New York, 1975, viii + 236 pp., $26.95.

It is very simple to define a martingale. If $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \ldots$ is an increasing sequence of sub σ-fields of the σ-field $\mathfrak{F}$ in a probability space $(\Omega, \mathfrak{F}, P)$, a sequence $\{x_n\}$ of real random variables is called *adapted* if x_n is $\mathfrak{F}_n$ measurable for $n = 1, 2, \ldots$. The adapted sequence is further called a supermartingale if $E(x_{n+1}|\mathfrak{F}_n) \leqslant x_n$, $n \geqslant 1$. It is called a martingale if the inequality is replaced by an equality and a submartingale if the inequality is reversed. Just one more definition is effectively all that is needed. A positive integer valued random variable ν is called a *stopping time* if the Ω set $\{\omega|\nu = n\}$ is in $\mathfrak{T}_n$ for $n \geqslant 1$.

The present work seems to be the first devoted exclusively to the study of martingales, which the author says "without any doubt constitute the mathematical technique at the base of modern probability". It is doubtful that a statement of such scope would find general agreement, and it is a fact that few authors of books on probability have given an extensive treatment. To the reviewer's knowledge only in the treatises of J. Doob **[1]**, who made fundamental contributions to the subject, M. Loève **[2]**, and L. Breiman **[3]** have martingales received an in-depth study. In Neveu's **[4]** earlier book there is a long section on the subject, and the present book seems to be an extension of this. Feller **[5]** has a long treatment, but abridged in that the basic convergence theorems are not given. There is little or nothing of the subject in the books by S. Karlin **[6]**, E. Parzen [7], B. Gnedenko **[8]**, K. L. Chung **[9]**, M. Rosenblatt **[10]**, A. Rényi **[11]**, Yu. Prokhorov-Yu. Rozanov **[12]**.

In the preface Neveu states that the material in the book has been used as a one semester course in the "third cycle" at the University of Paris. This corresponds roughly to the second year of graduate studies in American universities. A good course in measure theory is a prerequisite, as is a knowledge of the "basic notions of probability theory". In addition the development requires a knowledge of some aspects of linear space theory (including basic Hilbert space theory), although such things as Hölder's inequality are proved. Conditional expectations are defined via orthogonal projections in the Hilbert space $L^2(\Omega, \mathfrak{T}, P)$. Thus a well-prepared and motivated second year graduate student could undertake this material, but to finish it in one semester would require intensive work. Four of the nine chapters have problems at the end. These are of varying difficulty, but none is routine.

Here is a brief outline of the topics covered. After a preliminary chapter on conditional expectations, σ-fields, linear subspaces of the Banach spaces $L^p(\Omega, \mathfrak{T}, P)$, $p \geqslant 1$, and linear operators over them, a chapter is devoted to positive supermartingales. Here the situation is especially simple and striking–these martingales converge a.e., $n \to \infty$. A discussion follows concerning L^p convergence, $p \geqslant 1$ and the expectation of stopped martingales. There follows a chapter with five applications of the basic convergence theorem. (1) There is a proof of the Lebesgue decomposition of measures and the calculation of a specific Radon-Nikodym derivative by martingale convergence. (2) The necessary and sufficient conditions are deduced that two infinite products of independent probability measure spaces be mutually singular or equivalent, with applications to statistical likelihood ratios. (3) There is a proof that the Fourier series of a function $f \in L^p$, $p \geqslant 1$, associated with a Haar system converges in L^p and a.e. (4) There is a proof that every orthonormal expansion of an element of a Gaussian space converges a.e. (elementary Hilbert space theory gives the L^2 convergence). (5) Applied to Markov chains the study turns to positive superharmonic functions and the convergence theorem is utilized to find a criterion for a set of states to be recurrent.

Next the positivity requirement is dropped so that the conditions for a.e. convergence are more involved. The simplest such condition is

$$\limsup E(|x_n|) < \infty,$$

but this is not sufficient to ensure x_n converges in L^1, a condition that is important for the applications. Also important in applications are conditions ensuring that a stopped martingale preserves expectations. In this connection the author applies his results to likelihood ratios, obtaining Wald's identity and generalizations.

There is a chapter on extensions of martingales to families of random variables indexed by a directed family, to vector valued random variables, and to reversed martingales (adapted to a decreasing family of σ-fields).

Next there is an extensive treatment of the optimal stopping problem and its applications to Markov chains, random walks and sequential analysis. There is a chapter on Doob's decomposition of a submartingale into the sum of a martingale and an increasing process, and another on the decomposition of a supermartingale into the difference of a martingale and an increasing process. The first of these has important applications to square integrable martingales and leads, e.g., to a law of the iterated logarithm for such martingales. The second has an application to the quadratic variation of a martingale, and to the potential theory of Markov chains. There is a discussion of recent work on martingale transforms due to Burkholder, generalizing stopped martingales. A final appendix on the use of Young's functions in martingale theory examines the convergence of martingales in Orlicz spaces, generalizing earlier discussion of L^p convergence.

The exposition is first rate, as in earlier works of the author. It is clean, economical and elegant, with admirable directness in presenting the right theorems at the right places. It is well motivated without overselling the subject and what is often rare in the writings of French probabilists, it is accessible to the general public. It is a great pleasure to read casually or study intensively.

The only criticism the reviewer has is that it neglects the historical development of the subject and, with rare exceptions, does not attribute in the text authorship of the theorems quoted. Although there are over 300 items in bibliography there are not more than a half dozen references to them in the text. It surely might have been mentioned, e.g., that Paul Lévy was apparently the first to study martingales and to prove for them that the laws of large numbers, the central limit theorem and the law of the iterated logarithm hold. (Lévy did not know of the convergence theorems for martingales.) It is fashionable of late to downgrade the history and development of the subject treated by relegating it to a few notes at the end of the chapters, or the end of the book. This, in the reviewer's opinion, is bad enough, but should not be pushed so far as to omit it altogether.

The English edition includes some minor changes, a number of additional items in the bibliography, and a new section on Banach spaces of martingales with bounded root mean square variation and its dual, termed BMO.

A few more words are in order concerning the English edition of this fine book. The original publisher, Masson et Cie, is an old and honorable firm which takes evident pride in its product. The typescript is all hand set, the indices, sub and superscript are genuine, a sequence of parentheses varies in form for ease in reading, the type is clear and crisp and the paper is good grade glossy. The various parts of the text are set off in different type (e.g. the definitions are in capital letters) to help the reader. The English version suffers in comparison. The formulas have been set by machine to the maximum possible. The indices in sums, unions, lim sup, etc. appear as subscripts rather than under the corresponding signs. The successive parentheses are all of the same size and round shape and neither the paper nor the print is as good as in the French edition–there are several places where the impression of the type has been so feeble as to make the printing illegible. The only concession to variety in setting off parts of the text has been to italicize the propositions, theorems and corollaries. All of this slows down the reading and makes it less agreeable than in the original.

In addition the English text shows signs of hasty proofreading. There are numerous sub and superscripts which have wandered, have disappeared or been changed. One passage has been so garbled, with several lines missing, as to make it unintelligible without recourse to the original.[1] Additionally an unfortunate misnumbering in the original bibliography should have been, but was not, corrected.

The translation is good in the sense that it is literal and grammatical, and for the most part exact. But it cannot be said to be entirely satisfactory, and it was because of a number of rough passages that the reviewer turned to the French original. The translator is often too literal and incorporates too many "busy words" from the French that should not be translated. Without making a systematic survey the reviewer found a number of infelicities which deform the author's meaning. The phrase "nous rencontrerons aussi à plusieurs reprises des temps d'arrêt . . . " is given as "we will also meet many instances of stopping times . . . " rather than "we will also encounter several instances of stopping times . . . ". The phrase " . . . nous conviendrons de le poser égal à $+\infty$ . . . " is rendered " . . . it is convenient to put it equal to $+\infty$ . . . " rather than " . . . we agree to put it equal to $+\infty$ . . . ". The sentence "A cet effet nous etablirons un résultat beaucoup plus précis." is translated as "To this end we will establish an extremely precise result" rather than "To this end we shall establish a much more precise result". There are cases when the translation is misleadingly wrong. For example "nous supposerons dorénavant que . . . " is given as "we will suppose for the moment that . . . " rather than "we will suppose henceforth that . . . " and the translator goes so far as to translate "La primitive de cette fonction, soit $\int_0^t \varphi(s)\,ds$, est alors . . . " as "The indefinite integral of this function, say $\int_0^t \varphi(s)\,ds$, is then . . . " which is bound to confuse the student.

In general there is a certain lack of eveness in the English, in contrast with the French. The author, in common with other French writers, takes great care in making the language as smooth and precise as possible, and in this respect Neveu is impeccable in his lucidity. The French language is perhaps superior to English in the abstract, which is particularly an asset in probability theory.

If we add to these remarks the fact that the English edition costs over 40% more than the French, at 84 francs, it is legitimate to ask about the point of

[1]In the last paragraph of p. 30, first line, replace $\mathfrak{B}_n$ by $\mathfrak{B}_\infty$; delete the second line entirely and add instead of it "true for any r.v. $Z \geqslant 0$, not necessarily bounded, which is integrable in the pth power".

diminishing returns. It is probably hopeless to buck the current trend toward eliminating all requirements for foreign languages, and we can probably no longer ask that advanced graduate students have minimal competence in simple, elegant French. But we can perhaps ask that the translations serve the students as well as the original.

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Potential theory on locally compact abelian groups, by Christian Berg and Gunnar Forst, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 87, Springer-Verlag, Berlin, Heidelberg, New York, 1975, vi + 197 pp., $25.40.

There are not many books on the general potential theory from a nonprobabilistic point of view, and nearly none is concerned with convolution kernels different from the newtonian kernel (Landkoff's book being partly an exception). Therefore this book fills a gap and will be welcome and useful.

The authors consider only a simple and important case, where everything runs smoothly: the potential kernel is a *positive* convolution operator on a locally compact *abelian* group, and is the "vague" integral of a *transient* semigroup of positive measures. There is no mention of important and recent papers on nonabelian groups and recurrent semigroups. On the other hand probabilistic interpretations in terms of Hunt's processes are not given.

A good deal of the treated material has been well known for many years, but appears for the first time in a text-book (of course such a book should have been written before). The three following topics deserve a particular mention:

(a) *The study of negative definite functions* (the terminology is due to Beurling,