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THEORY OF ANNIHILATION GAMES

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Throughout, $R = (V(R), E(R))$ is a finite loopless digraph with vertex set $V(R)$ and edge set $E(R) \subset V(R) \times V(R)$, which may contain cycles. Let $F(u) = \{v \in V(R): (u, v) \in E(R)\}$, Z = nonnegative integers, $GF(2)^n$ = the n-fold cartesian product of $GF(2)$.

Put any number of stones on distinct vertices of R. Two players play alternately. Each player at his turn moves one stone from a vertex u to some $v \in F(u)$. If v was occupied, both stones get removed (*annihilation*). The player making the last move wins. If there is no last move, the game is a tie.

Such an *annihilation game* belongs to a large class of combinatorial games discussed in [1], [3], which are analyzable by the *Generalized Sprague-Grundy Function* (GSG-function) G: $V(R) \longrightarrow Z \cup \{\infty\}$ [1], [2], [3] with associated counter function c: $V^f(R) \longrightarrow Z$, where $V^f(R) = \{u \in V(R): G(u) < \infty\}$ [2]. Here R is the *game-graph* of the game.

Our main result is the construction of a complete strategy for the game, which is polynomial in $n = |V(R)|$.

Let $C(R)$ be the game-graph of the annihilation game on R, also called the *contrajunctive compound* of R. If $V(R) = \{u_1, \ldots, u_n\}$, the vertices of $V(C(R))$ (= game positions) constitute the set of all n-tuples $\bar{u} = (\alpha_1, \ldots, \alpha_n)$ over $GF(2)$, where $\alpha_i = 1$ if and only if a stone is on u_i. Also $(\bar{u}, \bar{v}) \subset E(C(R))$ if and only if there is a move from $\bar{u}$ to $\bar{v}$. Thus $V(C(R))$ is identical with the linear space $GF(2)^n$ under the operation $\oplus$, Σ' of *Nim-sum* (below: *Generalized Nim-sum* [1], [3]).

LEMMA 1. *Let*

$$C^f(R) = \{\bar{u} \in V(C(R)): G(\bar{u}) < \infty\}, \quad C_i(R) = \{\bar{u} \in V(C(R)): G(\bar{u}) = i < \infty\}.$$

Then

(i) *$C^f(R)$ is a linear subspace of $V(C(R))$.*

(ii) *G is a homomorphism from $C^f(R)$ onto $GF(2)^t$ with kernel $C_0(R)$ ($t = O(\log_2 n)$). In fact,*

$$G(\bar{u}) < \infty \Rightarrow G(\bar{u} \oplus \bar{v}) = G(\bar{u}) \oplus G(\bar{v}).$$

(iii) $\{C_i(R): 0 \leqslant i < 2^t\} = C^f(R)/C_0(R)$.

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Let $L_i^k(R) = \{\bar{u} \in C_i(R): |\bar{u}| = k\}$, $LF^k(R) = \{\bar{u} \in C^f(R): |\bar{u}| = k\}$, $\mathfrak{L}(S)$ = linear span of S, $\mathfrak{S}_0(R) = L_0^4(R) \cup L_0^2(R) \cup L_0^1(R)$, $\mathfrak{S}^f(R) = \mathfrak{S}_0(R) \cup LF^2(R)$.

LEMMA 2. (i) $C_0(R) = \mathfrak{L}(\mathfrak{S}_0(R))$.

(ii) $C^f(R) = \mathfrak{L}(\mathfrak{S}^f(R))$.

(iii) *There exists a basis* $\beta^f = (\bar{u}_1, \ldots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_t)$ *for* $C^f(R)$ *such that* $\beta_0 = (\bar{u}_1, \ldots, \bar{u}_m)$ *is a basis of* $C_0(R)$ *and* $\bar{v}_i \in L_{j(i)}^2(R)$, *where* $j(i) = 2^{i-1}$ $(1 \leqslant i \leqslant t)$.

Note. For $m \geqslant 0$, denote by $C^{(m)}(R)$ the subgraph of $C(R)$ with vertices $\bar{u}$ satisfying $|\bar{u}| \leqslant m$. Then $C^{(m)}(R)$ has $O(n^m)$ vertices, and $\bar{u} \in V(C^{(m)}(R)) \Rightarrow F(\bar{u}) \subset V(C^{(m)}(R))$. Hence G on $C^{(m)}(R)$ can be computed from $C^{(m)}(R)$ alone. In particular, $\mathfrak{S}^f(R) \subset V(C^{(4)}(R))$. Hence $\mathfrak{S}^f(R)$ can be computed in $O(n^6)$ steps using standard algorithms for computing the GSG-function [1].

THEOREM 1. *There exists an* $n \times n$ *matrix* Γ *over* $GF(2)$ *which can be computed polynomially, such that for every* $\bar{u} \in V(C(R))$ *we have* $\Gamma \cdot \bar{u}' = (\epsilon_1, \ldots, \epsilon_n)'$, where

$$\bar{u} = \sum_{i=1}^{m}{}' \epsilon_i \bar{u}_i \oplus \sum_{j=1}^{t}{}' \epsilon_{m+j} \bar{v}_j \oplus \sum_{k=1}^{n-m-t}{}' \epsilon_{m+t+k} \bar{z}_k,$$

and $(\bar{z}_1, \ldots, \bar{z}_k)$ *is a basis of a complementary space of* $C^f(R)$. *Moreover, letting* $Q(\bar{u}) = (\epsilon_n, \epsilon_{n-1}, \ldots, \epsilon_{m+1})$, Q *is a homomorphism from* $V(C(R))$ *onto* $GF(2)^{n-m}$ *with kernel* $C_0(R)$, *such that* $G(\bar{u}) = \bar{Q}(\bar{u}) = \Sigma_{i=1}^{\prime n} \epsilon_{m+i} 2^{i-1}$ *if* $(\epsilon_n, \epsilon_{n-1}, \ldots, \epsilon_{m+t+1}) = (0, 0, \ldots, 0)$; $G(\bar{u}) = \infty$ *otherwise.*

CONCLUSION 1. The N, P, T classification [1], [2], [3] and the GSG-function of any $\bar{u} = (\alpha_1, \ldots, \alpha_n)$ can be computed polynomially. In particular, the values $\bar{Q}(\bar{u}_i)$, where $\bar{u}_i = (\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i = 1$, $\epsilon_j = 0$ $(j \neq i; i = 1, \ldots, n)$, determine $G(\bar{u})$. Indeed,

$$Q(\bar{u}) = \sum_{\alpha_i = 1}{}' Q(\bar{u}_i) = (\delta_n, \delta_{n-1}, \ldots, \delta_{m+1}),$$

and so $G(\bar{u}) = Q(\bar{u})$ if $\delta_n = \cdots = \delta_{m+t+1} = 0$; $G(\bar{u}) = \infty$ otherwise. This, however, does not yet guarantee the *realization* of a winning strategy, because of possible cycling.

Let $\bar{u} \in P = C_0(R)$. Then $\bar{u}$ has a *representation* $\tilde{u} = (\bar{y}_1, \ldots, \bar{y}_k) \subset \mathfrak{S}_0(R)$ $(k \leqslant n)$ in the sense that $\bar{u} = \Sigma_{i=1}^{\prime k} \bar{y}_i$. For example, initially we may have $\tilde{u} \subset \beta_0$. Let c be a *monotonic* counter function on $C^{(4)}(R)$ (i.e., $G(\bar{u}) < G(\bar{v}) \Rightarrow c(\bar{u}) < c(\bar{v})$). We can choose $c(\bar{u}) = O(n^4)$ for all $\bar{u} \in V(C^{(4)}(R))$. Let $\tilde{c}(\tilde{u}) = \Sigma_{i=1}^{k} c(\bar{y}_i)$. Then $\tilde{c} = O(n^5)$.

THEOREM 2. *There is a function* Λ_0 *which can be computed polynomially, such that for every* $\bar{u} \in C_0(R)$ *and every* $\bar{v} \in F(\bar{u})$,

$$\Lambda_0(\tilde{u}, \bar{v}) = \tilde{w} = (\bar{w}_1, \ldots, \bar{w}_k) \subset \mathfrak{S}_0(R),$$

$$\bar{w} = \sum_{i=1}^{k}{}' \bar{w}_i \in F(\bar{v}) \cap P,$$

$$\tilde{c}(\tilde{w}) < \tilde{c}(\tilde{u}).$$

Note. The representation $\tilde{w}$ is obtained from $\tilde{u}$ in a bounded number of transformations. Details are omitted.

CONCLUSION 2. Using Λ_0 and starting from any N-position, every annihilation game can be won in $O(n^5)$ moves using polynomial computation time throughout. The function Λ_0 implies a *winning strategy in the wide sense* [3]. A bounded number of cycles may be traversed in realizing the strategy (but no cycling takes place in the "representation space"). We do not know if a *winning strategy in the narrow sense* exists which is always polynomial.

Further results, ramifications and proofs will appear elsewhere.

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