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BORDISM OF DIFFEOMORPHISMS

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1. **Introduction.** In this note we determine the bordism groups Δ_n of orientation preserving diffeomorphisms of n-dimensional closed oriented smooth manifolds. These groups were introduced by W. Browder [1]. Winkelnkemper showed that each diffeomorphism of the sphere S^n is nullbordant [7]. On the other hand, he showed that Δ_{4k+2} is not finitely generated. Medrano generalized this result to Δ_{4k} [5]. For this he introduced a powerful invariant in the Witt group $W_\pm(\mathbf{Z}, \mathbf{Z})$ ($I_\pm$ in Medrano's notation) of isometries of free finite-dimensional $\mathbf{Z}$-modules with a symmetric (antisymmetric) unimodular bilinear form. The invariant is given by the middle homology modulo torsion, the intersection form and the isometry induced by the diffeomorphism. For a diffeomorphism f: $M \longrightarrow M$ we denote this invariant by $I(M, f)$, the isometric structure of (M, f). It is a bordism invariant and leads to a homomorphism I: $\Delta_{2k} \longrightarrow W_{(-1)^k}(\mathbf{Z}, \mathbf{Z})$.

Neumann has shown that the homomorphism I is surjective, that $W_\pm(\mathbf{Z}, \mathbf{Z}) \otimes \mathbf{Q} \cong \mathbf{Q}^\infty$ and that $W_\pm(\mathbf{Z}, \mathbf{Z})$ contains infinitely many summands of orders 2 and 4 [6]. On the other hand, $W_\pm(\mathbf{Z}, \mathbf{Z})$ is a subgroup of $W_\pm(\mathbf{Z}, \mathbf{Q})$, the Witt group of isometries of finite-dimensional $\mathbf{Q}$-vector spaces. This group plays an important role in the computation of bordism groups C_{2k-1} of odd-dimensional knots, which can be embedded in $W_{(-1)^k}(\mathbf{Z}, \mathbf{Q})$. It is known that $W_\pm(\mathbf{Z}, \mathbf{Q}) \cong \mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$ [3]. Thus the group $W_\pm(\mathbf{Z}, \mathbf{Z})$ is also of the form $\mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$.

It turns out that the isometric structure is essentially the only invariant for bordism of diffeomorphisms.

2. **Bordism of odd-dimensional diffeomorphisms.** Two diffeomorphisms (M_1, f_1) and (M_2, f_2) are called bordant if there is a diffeomorphism (N, F) on an oriented manifold with boundary such that $\partial(N, F) = (M_1, f_1) + (-M_2, f_2)$. The bordism classes $[M^n, f]$ form a group under disjoint sum, called Δ_n.

The mapping torus of a diffeomorphism (M, f) is $M_f = I \times M/(0, x) \sim (1, f(x))$. This construction leads to a homomorphism $\Delta_n \longrightarrow \Omega_{n+1}$ ($[M, f] \longmapsto [M_f]$), where Ω_{n+1} is the ordinary bordism group of oriented manifolds.

In [4] we proved the following result.

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THEOREM 1. *For $k > 2$ the map $[M, f] \mapsto ([M], [M_f])$ is an isomorphism $\Delta_{2k-1} \longrightarrow \Omega_{2k-1} \oplus \hat{\Omega}_{2k}$, where $\hat{\Omega}_{2k}$ is the kernel of the signature homomorphism τ.*

3. **The even-dimensional case.** Consider triples $(G, \langle\ ,\ \rangle, h)$, where G is a finite-dimensional free Z-module, $\langle\ ,\ \rangle$ a symmetric (resp. antisymmetric) unimodular bilinear form on G and h an isometry of $(G, \langle\ ,\ \rangle)$. $(G, \langle\ ,\ \rangle, h)$ is called hyperbolic if there exists an invariant subkernel, i.e. a subspace $U \subset G$ with $U \subset U^{\perp}$, $2 \dim U = \dim G$ and $h(U) \subset U$. $(G, \langle\ ,\ \rangle, h)$ and $(G', \langle\ ,\ \rangle', h')$ are called bordant if $(G, \langle\ ,\ \rangle, h) \oplus (G', -\langle\ ,\ \rangle', h')$ is hyperbolic. This is an equivalence relation. The equivalence classes form a group under orthogonal sum, called $W_+(\mathbf{Z}, \mathbf{Z})$ (resp. $W_-(\mathbf{Z}, \mathbf{Z})$).

The isometric structure of a diffeomorphism (M^{2k}, f) is given by $(H_k(M; \mathbf{Z})/\mathrm{Tor}, \circ, f_*)$, where $\circ$ is the intersection form. If (M, f) bounds a diffeomorphism (N, F) the isometric structure is hyperbolic, an invariant subkernel being given by the kernel of i_*: $H_k(M; \mathbf{Z})/\mathrm{Tor} \longrightarrow H_k(N; \mathbf{Z})/\mathrm{Tor}$, so we have a homomorphism I: $\Delta_{2k} \longrightarrow W_{(-1)^k}(\mathbf{Z}, \mathbf{Z})$. Neumann has shown that this homomorphism is surjective [6].

THEOREM 2. *For $k > 1$ the homomorphism*

$$\Delta_{2k} \longrightarrow W_{(-1)^k}(\mathbf{Z}, \mathbf{Z}) \oplus \hat{\Omega}_{2k} \oplus \Omega_{2k+1},$$

$$[M, f] \mapsto (I(M, f), [M] - \tau(M)[P_k\mathbf{C}], [M_f])$$

is an isomorphism (k even), injective with cokernel $\mathbf{Z}_2$(k odd).

4. **Idea of the proof.** Consider a diffeomorphism (M^n, f) such that M and M_f are nullbordant. This implies that there exists a differentiable map g: $N^{n+2} \longrightarrow S^1$ with $\partial N = M_f$ and $g|_{\partial N}$ the canonical projection from M_f to S^1. Let $x \in S^1$ be a regular value of g. $F := g^{-1}(x)$ is a 1-codimensional submanifold of N with trivial normal bundle meeting ∂N transversally along ∂F: Cut N along F to obtain a differentiable manifold N_F with corners. The boundary of N_F consists of two copies F_0 and F_1 of F with opposite orientations and a manifold $V = \partial N_{\partial F}$ fibred over the unit interval I. $\partial V = \partial F_0 + \partial F_1$. The corners of N_F are at ∂F_0 and ∂F_1.

We now make the following strong assumption (compare [2, 2.3]): The components of F are simply connected and F_0 and F_1 are deformation retracts of N_F.

Then $(N_F; F_0, F_1)$ is a relative h-cobordism between $(F_0, \partial F_0)$ and $(F_1, \partial F_1)$. For $n > 5$ the h-cobordism theorem implies that the diffeomorphism on M can be extended to a diffeomorphism of F.

Our aim is, starting with an arbitrary N_F to obtain an $N'_{F'}$ which satisfies the above assumption. For this we modify N_F by addition and subtraction of handles. If n is odd ($n \neq 3$) we have shown in [4] that this works. If n is even

($n > 2$) and $I(M, f) = 0$ we can do the same. The details of the proof will appear elsewhere.

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