BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 5, September 1976

# TRANSFERENCE RESULTS FOR MULTIPLIER OPERATORS

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Communicated by François Treves, January 30, 1976

The purpose of this paper is to show a transference result of the type obtained in [4] and [5] for convolution operators acting on functions defined on $\Sigma_{n-1}$, the unit sphere of $\mathbf{R}^n$. As a consequence we obtain a multiplier theorem for expansions in spherical harmonics and Gegenbauer polynomials. Also, Zygmund's inequality for Cesàro sums and that for Littlewood-Paley function $g_\delta$, due to Bonami and Clerc [1], are easily obtained using our results [6]. I wish to express my appreciation to my Ph.D advisors, Professor R. Coifman and G. Weiss, for their encouragement and help in the preparation of this work.

**Introduction.** Let SO($n$) be the group of all rotations of $\mathbf{R}^n$. The left regular representation of SO($n$) defined by $R_u f(x) = f(u^{-1}x)$, $u \in \mathrm{SO}(n)$ and $f \in L^2(\Sigma_{n-1})$, decomposes into a direct sum of finite dimensional irreducible representations $R^k$ ($n \geqslant 3$), $k = 0, 1, \ldots$. $L^2(\Sigma_{n-1}) = \Sigma_{k=0}^\infty H_k$, where $H_k$, the space of the representation $R^k$, consists of the spherical harmonics of degree $k$ [2], [8], [9]. If $f \in L^2(\Sigma_{n-1})$, $f(x) = \Sigma_{k=0}^\infty (Z_{e,n-1}^{(k)} * f)(x)$, where $Z_{e,n-1}^{(k)}(x)$ is the zonal spherical harmonic of degree $k$ and pole $e = (0, \ldots, 0, 1)$ and $*$ denotes convolution on $\Sigma_{n-1}$. A multiplier $M$, is an operator that commutes with the action of SO($n$) on $\Sigma_{n-1}$ and is defined on the class $P$ of finite linear combinations of elements in the spaces $H_k$. Such $M$ assume the form

$$Mf(x) = \sum m_k (Z_{e,n-1}^{(k)} * f)(x) \quad \text{(finite sum)}.$$

**Multipliers for expansions in spherical harmonics.** Let $H$ be a Hilbert space over the complex numbers and let $L^p(\Sigma_{n-1}, H)$, $1 \leqslant p \leqslant \infty$, be the space of functions $f\colon \Sigma_{n-1} \longrightarrow H$ defined in the usual way replacing absolute values by $\|\cdot\|_H$. For the left regular representation of SO($n$) on $L^2(\Sigma_{n-1}, H)$ we have a decomposition entirely similar to the one described above [3]. To a bounded operator on $L^2$ which commutes with rotations,corresponds a bounded sequence $\{m_k\}_{k=0}^\infty$ of operators on $H$ such that $Mf(x) = \Sigma\, m_k (Z_{e,n-1}^{(k)} * f(x))$ (finite sum) for every $f \in P$. The operator valued function

$$K_r(x) = \sum_{k=0}^\infty r^k Z_{e,n-1}^{(k)}(x) m_k, \qquad r \in [0, 1),$$

is continuous. We write $Mf(x) = \lim_{r \to 1} (K_r * f)(x)$.

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*AMS (MOS) subject classifications* (1970). Primary 42A18, 43A85.

THEOREM 1. *Let $M_r$ be defined on the class $P$ on $\Sigma_{n-2}$ by letting*

$$M_r g(x) = [(|\sin\theta| K_r(\theta)) * g](x),$$

*where $r \in [0, 1)$ and $\theta$ is the angle between a variable in $\Sigma_{n-1}$ and $e$.*
*If $M_r$ is bounded uniformly for $r$ close to 1, i.e.*

$$\int_{\Sigma_{n-2}} \|M_r g(x)\|_H^p \, dx \leqslant A_p^p \int_{\Sigma_{n-2}} \|g(x)\|_H^p \, dx,$$

*$1 \leqslant p < \infty$ and $A_p$ is a constant depending only on $p$, then*

$$\int_{\Sigma_{n-1}} \|Mf(x)\|_H^p \, dx \leqslant A_n^p A_p^p \int_{\Sigma_{n-1}} \|f(x)\|_H^p \, dx.$$

Let $f \in L^1(\mathrm{SO}(n))$; then

$$\int_{\mathrm{SO}(n)} f(u)\, du = c_n \int_{\mathrm{SO}(n-1)} \int_{\mathrm{SO}(n-1)} \int_0^{2\pi} f(\sigma a(\theta)\sigma') |\sin\theta|^{n-2} \, d\theta \, d\sigma \, d\sigma'$$

where $du$ and $d\sigma$, $d\sigma'$ are the Haar measures of $\mathrm{SO}(n)$, $\mathrm{SO}(n-1)$ respectively, and $a(\theta)$ is a rotation by the angle $\theta$ in the subspace of $\mathbf{R}^n$ generated by the vectors $e$ and $(0, \ldots, 0, 1, 0)$. Using (2) and the methods of [4] and [5] one obtains the above result.

THEOREM 3. *Let $\{K_j\}_{j=0}^{\infty}$ be a sequence of integrable zonal functions on $\Sigma_{n-1}$. Define the maximal operator $K^* f(x) = \sup_j \|(K_j * f)(x)\|_H$ on $L^p(\Sigma_{n-1}, H)$. If the maximal operator $k^* g(x) = \sup_j \|(|\sin\theta| K_j(\theta)) * g(x)\|_H$ is bounded on $L^p(\Sigma_{n-2}, H)$ with operator norm $B_p$, then $K^*$ is bounded also and its norm bounded by $A_n B_p$.*

When $H$ is the field of complex numbers we obtain

THEOREM 4. *Let $N = [n/2]$. If the sequence $\{\mathcal{D}^N(m_k)\}_{k=0}^{\infty}$ defines a bounded multiplier on $L^p(\Sigma_1)$, $1 \leqslant p < \infty$, then $\{m_k\}_{k=0}^{\infty}$ defines a bounded multiplier on $L^p(\Sigma_n)$, where $\mathcal{D}(m_k) = k m_k - (k-2) m_{k-2}$ and $\mathcal{D}^t(m_k) = \mathcal{D}(\mathcal{D}^{t-1}(m_k))$.*

The Marcinkiewicz multiplier theorem [7], together with the above result, give us a multiplier theorem that contains that of Bonami and Clerc [1].

**Multipliers for expansions in Gegenbauer polynomials.** $L_\lambda^p(-1, 1)$ denotes the space of complex valued measurable functions $f$ on $[-1, 1]$ with respect to the measure $dm_\lambda(x) = (1 - x^2)^{\lambda - 1/2}\, dx$, where $\lambda > 0$ and $dx$ is Lebesgue measure. To each $f \in L_\lambda^p$, we associate the formal sum $f(x) \sim \Sigma_{k=0}^{\infty} c_k \hat{f}(k) \mathrm{C}_k^\lambda(x)$, where $\mathrm{C}_k^\lambda(x)$ is the normalized Gegenbauer polynomial of order $\lambda$, $\mathrm{C}_k^\lambda(1) = 1$, $\hat{f}(k)$ the Fourier coefficient and $c_k^{-1} = \|\mathrm{C}_k^\lambda\|_{2,\lambda}^2$.

A multiplier $M$ assumes the form $Mf(x) \sim \Sigma_{k=0} m_k c_k \hat{f}(k) \mathrm{C}_k^\lambda(x)$, where $\{m\}_{k=0}^{\infty}$ is a sequence of complex numbers.

THEOREM 5. *Let* $\lambda$, $\delta$ *be positive real numbers. If the convolution operator with kernel* $g(y)(1-y^2)^\delta$ *is bounded on* $L^p_\lambda$ *with operator norm* $A_{p,\lambda}$, *then* $g(y)$ *defines a bounded convolution operator on* $L^p_{\lambda+\delta}$ *with norm bounded by* $C_{\beta,\delta}A_{p,\lambda}$.

This theorem implies a transference result for multipliers similar to Theorem 1

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