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NORMALITY VERSUS COUNTABLE PARACOMPACTNESS IN PERFECT SPACES

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Introduction. The purpose of this announcement is to present, in a unified fashion, solutions to long outstanding questions concerning the relationship between countable paracompactness and normality conditions in perfect spaces. Each section of this paper is the contribution of a single author and is so designated.

It was established in 1951 by Dowker [4] that in perfect spaces (i.e. spaces in which closed sets are G_δ-sets), normality implies countable paracompactness. However, the validity of the converse has remained an open question until the present. In particular, the relationships between normality, countable paracompactness, and pseudo-normality in Moore spaces has been of considerable interest ([8], [10], [11], [17], [19], and [20], for example). In this paper, the authors (1) produce an example of a countably paracompact, perfect, nonnormal T_3-space, (2) produce an example of a pseudo-normal, separable, noncountably paracompact Moore space, and (3) show the consistency and independence of the existence of a countably paracompact, separable, nonnormal Moore space. In addition, several corollaries are given which answer open questions concerning the hereditary and mapping properties of countable paracompactness in perfect spaces.

I. **(Wage [17]).** The construction given below associates a regular, nonnormal T_2-space X^* to each normal, noncollectionwise normal space X.

THE MACHINE. Suppose X is a normal T_2-space and $\{H^\alpha\}_{\alpha<\lambda}$ is a discrete collection of closed sets which cannot be separated by open sets. Let $D = X\backslash H$, where $H = \bigcup\{H^\alpha: \alpha < \lambda\}$. Denote

$$X^* = (X \times \{0, 1\}) \cup (D \times \{(\alpha, \beta): \alpha, \beta < \lambda \text{ and } \alpha \neq \beta\}).$$

For each $A \subset X$ and $\delta \in \{0, 1\} \cup \{(\alpha, \beta): \alpha, \beta < \lambda \text{ and } \alpha \neq \beta\}$, let A_δ denote $(A \times \{\delta\}) \cap X^*$. Now, define a base $\mathcal{B}$ for the desired topology on X^* as follows

(1) if $x \in X^*\backslash(H_0 \cup H_1)$, let $\{x\} \in \mathcal{B}$; and

(2) if U is an open set in X and $\alpha < \lambda$ such that $U \subset (H^\alpha \cup D)$, let

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$$(\bigcup\{U_{(\alpha,\beta)}: \ \alpha \neq \beta < \lambda\} \cup U_0) \in \mathcal{B}$$

and let

$$(\bigcup\{U_{(\beta,\alpha)}: \ \alpha \neq \beta < \lambda\} \cup U_1) \in \mathcal{B}.$$

Note that X_0 and X_1 are two mutually exclusive closed sets in X^* which cannot be separated by open sets, hence X^* is not normal.

NOTATION. (CH) will denote the assumption of the continuum hypothesis and ($\neg$CH) will denote the assumption of its negation. The assumption of Martin' Axiom (see [16] and [17]) will be denoted by (M. A.).

THEOREM 1. *If X is a normal, noncollectwise normal T_2-space then X^* is a regular, nonnormal T_2-space and*

(1) *X perfect $\longrightarrow X^*$ perfect,*

(2) *X first countable $\longrightarrow X^*$ first countable,*

(3) *X Moore $\longrightarrow X^*$ Moore,*

(4) *X collectionwise Hausdorff $\longrightarrow X^*$ collectionwise Hausdorff,*

(5) *X countably paracompact $\longrightarrow X^*$ countably paracompact.*

COROLLARY 1. *There exists a perfect T_3-space which is countable paracompact but not normal.*

PROOF. Example H of [1] is a perfectly normal T_2-space that is not collectionwise normal. Hence, Theorem 1 yields the desired example.

COROLLARY 2. *If there exists a normal nonmetrizable Moore space, then there exists a countably paracompact, nonnormal Moore space.*

COROLLARY 3 (M.A. + $\neg$CH). *There exists a countably paracompact Moore space that is not normal.*

COROLLARY 4 (M.A. + $\neg$CH). *There exists a closed continuous mapping from a countably paracompact Moore space onto a T_2-space which fails to be countably paracompact.*

COROLLARY 5 (M.A. + $\neg$CH). *There exists a countably paracompact Moore space that is not hereditarily countably paracompact.*

Corollaries 4 and 5 follow from Theorem 1 and results by Zenor [**20**].

THEOREM 2 ($\diamondsuit$). *There exists a nonnormal T_3-space which is countably compact, perfect, first countable, locally compact, locally countable, zero-dimensional, and hereditarily separable.*

The construction of the example in Theorem 2 depends heavily on Jensen's $\diamondsuit$ and the technique used by Ostaszewski [9].

II. **(Fleissner [5]).**

JONES' LEMMA ([7]). *If S is a normal T_1-space, D is a dense subset of S, and Y is a closed discrete subset of S, then $2^{|Y|} \leqslant 2^{|D|}$.*

LEMMA F. *If S is a countably paracompact T_1-space, D is a dense subset of S, and Y is a closed discrete subset of S, then $|Y| < 2^{|D|}$.*

THEOREM 3 (CH). *Each countably paracompact, separable Moore space S is metrizable.*

PROOF. By Lemma F, each uncountable subset of S has a limit point. Each Moore space with this property is metrizable [7].

III. **(Reed [12]).** Using the basic splitting concept of the construction due to Wage in §I, the author of this section was able to construct the examples below which are remarkable for their simplicity.

EXAMPLE 1 (M.A. + $\neg$CH). There exists a countably paracompact separable Moore space which is not normal.

REMARK. Note that Example 1, together with Theorem 3, establishes that the existence of a countably paracompact, separable Moore space is consistent with and independent of the usual axioms of set theory.

EXAMPLE 2 (M.A. + $\neg$CH). There exists a countably paracompact, screenable Moore space which is not normal.

PSEUDO-NORMALITY. Proctor **[10]** defined a space to be pseudo-normal provided each two mutually exclusive closed sets, one of which is countable, can be separated by open sets. Proctor also noted that countably paracompact T_3-spaces are pseudo-normal and gave an example of a pseudo-normal, separable, nonmetrizable Moore space. Another such example was given by Tall **[17]**. However, to show the noncountable paracompactness of these spaces, one must assume the Continuum Hypothesis. In particular, under M.A. + $\neg$CH, Tall's example is known to be normal. An example of a pseudo-normal, noncountably paracompact, nonseparable Moore space was given in **[11]**.

EXAMPLE 3. There exists a pseudo-normal, separable Moore space which is not countably paracompact.

REMARKS ON THE CONSTRUCTIONS. A Q-set (respectively, λ-set) is an uncountable subset of the real line in which each subset (respectively, each countable subset) is a relative G_δ-set. Q-sets exist under M.A. + $\neg$CH **[16]** and λ-sets exist without any extra set-theoretic assumptions beyond the Axiom of Choice [7]. The construction of Example 1 is accomplished by considering a tangent disk space defined on a Q-set and splitting the Q-set into two disjoint copies which cannot be separated. The construction of Example 2 is done in a similar manner by considering Heath's "V" space **[6]** defined on a Q-set. Example 3 is constructed by considering a tangent disk space defined on a λ-set and splitting the λ-set into countably infinitely many copies. These constructions differ from the construction given in §I in that the space is not made Hausdorff by splitting points in the upper plane but by assigning disjoint neighborhoods in the same plane to corresponding points on the disjoint copies of the real line.

In [13], it is noted that Example 1 is neither continuously semimetrizable nor submetrizable, while Example 2 (in fact, a space such as Example 2 constructed without any extra set-theoretic assumptions) is continuously semimetrizable but not submetrizable. These examples thus answer questions raised in [2], [3], and [19] concerning the relationships between continuous semimetrizability, submetrizability, and a zero-set diagonal. Recall that in [14], it was shown that each normal Moore space of cardinality $\leqslant c$ is submetrizable. Also, the construction technique of Example 2 produces the first example of a perfect map from a screenable Moore space onto a nonscreenable Moore space.

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