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SINGULAR HAMMERSTEIN EQUATIONS AND MAXIMAL MONOTONE OPERATORS

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Consider the nonlinear integral equation of Hammerstein type

$$u(x) + \int_\Omega k(x, y) f(y, u(y)) \beta(dy) = h(x) \qquad (x \in \Omega), \tag{1}$$

where h and the solution u lie in a space X of measurable functions on Ω. The Hammerstein equation is said to be regular if for

$$F(u)(y) = f(y, u(y)) \qquad (y \in \Omega); \qquad Kv(x) = \int_\Omega k(x, y) v(y) \beta(dy) \qquad (x \in \Omega), \tag{2}$$

the operator KF is defined on all of X, and singular otherwise.

In some recent papers (summarized in [2]), the writers have studied the existence theory for regular Hammerstein equations in $L^p(\beta)$ with $1 < p \leqslant +\infty$ under very general assumptions on K and F. In later papers (cf. [4]), one of the writers has obtained general existence results for the singular case, using measure-theoretic arguments and mild compactness assumptions on K. We present results here without compactness assumptions based on a new theorem on linear monotone operators.

THEOREM 1. *Let X be a reflexive Banach space, L_0 and L_1 linear monotone mappings from X into 2^{X^*} with $L_0 \subseteq L_1^*$. Then there exists a maximal monotone linear map from X into 2^{X^*} such that $L_0 \subseteq L \subseteq L_1^*$.*

For single-valued, densely defined maps in Hilbert space, this coincides with a theorem of R. S. Phillips [6] obtained using ideas of M. Kreĭn [5]. For reflexive Banach spaces, in general, we have as a corollary a result obtained in 1968 by one of the writers [1]:

THEOREM 2. *Let X be a reflexive Banach space, L a closed linear monotone map from X into 2^{X^*}. Then L is maximal monotone if and only if L^* is monotone.*

We sketch the proof of Theorem 1 (detailed proofs are given in [3]). By a Zorn's Lemma argument we may construct a monotone linear map L with $L_0 \subseteq L \subseteq L_1^*$ such that L is maximal monotone *in the graph of* L_1^*. Let J be a duality map of X into X^* corresponding to a norm on X with X and X^* locally uniformly convex.

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Let w_0 be any element of X^*. It suffices to find u_0 in X such that $w_0 \in (L + J)(u_0)$. For each finite-dimensional subspace M of X, let ξ_M be the injection map of M into X, $\xi_M^*: X^* \longrightarrow M^*$. We form linear monotone mappings L_M and $L_{1,M}$ of M into 2^{M^*} with $L_M \subseteq (L_{1,M})^*$ by

$$L_M(x) = \xi_M^*(L(x)), \qquad L_{1,M}(x) = \xi_M^*(L_1(x)).$$

We apply the multivalued finite-dimensional version of Phillips' theorem (a simple direct proof for which is given in [3]) to obtain a maximal monotone mapping K_M from M to 2^{M^*} such that $L_M \subseteq K_M \subseteq (L_{1,M})^*$. Hence, we may find u_M in M such that $\xi_M^*(w_0) \in K_M(u_M) + \xi_M^*(J(u_M))$.

For each $[u, w]$ in $G(L_1)$ and for each $[x, y]$ in $G(L)$ with u and x in M,

$$\langle w_0 - J(u_M), u\rangle = \langle w, u_M\rangle, \tag{3}$$

$$\langle y + J(u_M) - w_0, x - u_M\rangle \geqslant 0. \tag{4}$$

The elements $\{ [u_M, J(u_M)] \}$ are bounded since J is coercive. Since X is reflexive, we may assume a filter $\{ [u_M, J(u_M)] \}$ converging weakly to $[u_0, y_0]$ in $X \times X^*$. Since equality (3) holds eventually for each $[u, w]$ in $G(L_1)$, we may take the limit to find that $\langle w_0 - y_0, u\rangle = \langle w, u_0\rangle$ for all $[u, w]$ in $G(L_1)$. Hence $[u_0, w_0 - y_0]$ lies in $G(L_1^*)$. From inequality (4) which holds eventually for each $[x, y]$ in $G(L)$, we obtain

$$\overline{\lim}\langle J((u_M), u_M)\rangle - \langle y_0, u_0\rangle \leqslant \langle y + y_0 - w_0, x - u_0\rangle. \tag{5}$$

Since J is pseudo-monotone, the left side is nonnegative. Since $[u_0, w_0 - y_0] \in G(L_1^*)$ and L is assumed maximal monotone in $G(L_1^*)$, $[u_0, w_0 - y_0]$ lies in $G(L)$. Replacing $[x, y]$ by this element, it follows that the left side of (5) is zero, and hence $y_0 = J(u_0)$. Thus $w_0 - J(u_0) \in L(u_0)$, i.e. $w_0 \in (L + J)(u_0)$. Q.E.D.

The application to singular Hammerstein equations is made through the following more general theorem:

THEOREM 3. *Let β be a finite measure on Ω, X a reflexive Banach space with $L^\infty(\beta) \subseteq X \subseteq L^1(\beta)$, $L^\infty(\beta) \subseteq X^* \subseteq L^1(\beta)$. Let F be a hemicontinuous, monotone angle-bounded map of X into X^* with $0 \in \operatorname{Int}(R(F))$. Let K be a bounded linear map of $L^1(\beta)$ into $L^1(\beta)$ with $\langle Kv, v\rangle \geqslant 0$ for all v in $L^\infty(\beta)$ Then for each h in X, there exists u in X such that $u + KF(u) = h$ and $\langle Kv - KF(u), v - F(u)\rangle \geqslant 0$ for all $v \in L^\infty(\beta)$ with $Kv \in X$.*

To prove Theorem 3, we may set $h = 0$ by a change of variables. Let L_1 be the mapping from X^* to X with effective domain $L^\infty(\beta)$ and with $L_1(v) = K'(v)$ where K': $L^\infty(\beta) \longrightarrow L^\infty(\beta)$ is the dual of K. Then L_1 is monotone and L_1^* is a restriction of K. Let $K^\#$ be the mapping from X^* to X with domain $D(K^\#) = \{ v \in L^\infty(\beta)$ and $Kv \in X\}$ and $K^\# v = Kv$. Since $K^\# \subset L_1^*$ we may find by Theorem 1, a maximal monotone operator L satisfying $K^{\#} \subseteq L \subseteq L_1^*$. Finally one solves $0 \in L^{-1}(u) + F(u)$.

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