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THE REGULARITY OF ELLIPTIC AND PARABOLIC FREE BOUNDARIES

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In this report we shall sketch a proof of the fact certain free boundaries in R^n are smooth. This result can be applied to the variational solutions of obstacle problems, filtration problems and the one phase Stefan problem for melting ice, which have recently been obtained by various authors [1], [3], [4].

We first consider the following localization of the weak solution of an elliptic free boundary problem: We are given an open set W, a linear elliptic operator $Au = \Sigma a_{ij}(x)\partial_i\partial_j$ ($a_{ij} \in C^3$ in a neighborhood of $\overline{W}$), and a function $v \in C^{1,1}(W)$ and satisfying: (1) $v \geqslant 0$, $Av = f$, where f has a C^α ($\alpha > 0$) extension f^* to a neighborhood of $\overline{W}$, with $f^* \geqslant \lambda > 0$; (2) $\partial W = \partial_1(W) \cup \partial_2(W)$ where $\partial_1 W$ is open in ∂W and $v = |\nabla v| = 0$ on $\partial_1 W$. ($\partial_1 W$ is a part of the free boundary.) F will denote an open subset of $\partial_1 W$ with $\overline{F} \subset \partial_1 W$.

THEOREM 1. *If $X_0 \in F$ is a nonzero density point for the complement CW of W, there is a ball $B_\rho(X_0)$ of radius ρ, centered at X_0, such that $F \cap B_\rho(X_0)$ is the graph of a C^1 function and $v \in C^2((W \cup F) \cap B_\rho(X_0))$.*

REMARK. This result has two virtues: first it shows that the variational solution is a classical one; second, it then follows from unpublished results of D. Kinderlehrer and L. Nirenberg that the gradient of the free boundary (as the graph of a function) is as differentiable as f and a_{ij}.

The proof goes as follows: First, Lemma 1, we prove that the pure second derivatives, v_{ii}, of v do not remain negative near F. More precisely, for $X \in W$, $v_{ii}(X) > -C|\log d(X, F)|^{-\epsilon}$, where $d(X, F)$ is the distance from X to F and $\epsilon > 0$. The geometric consequence of this fact is that, if $Y \in W$, $v(Y) > \rho^2$, and $d(Y, F) < \rho^{1/2}$, then there exists a half ball,

$$HB(Y, C\rho|\log \rho|^{\epsilon'}) = B(Y, C\rho|\log \rho|^{\epsilon'}) \cap \{X: \langle X - Y, \eta\rangle \geqslant 0\}$$

which is contained in W. If we recall that for $X \in \overline{W}$, $\sup_{B_\rho(X)\cap W} v \geqslant C\rho^2$, this lemma provides for each $X \in \overline{W}$ a half ball contained in W, whose radius is much larger than the distance between its center and X.

The following step, Lemma 2, controls how rapidly CW must become

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"thin" at a point X_0 of F of zero density with respect to $\mathcal{C}W$: Let $D(X_0, \epsilon_0, \rho, \eta)$ denote the spherical cap $\{X: |X - X_0| = \rho, \alpha(X - X_0, \eta) < \epsilon_0\}$ (η a unit vector, $\alpha(U, V)$ the angle between U and V, $\epsilon_0 < C$ depending on the ellipticity of A). There exists a $\rho_0(\epsilon_0)$, such that if $D(X_0, \epsilon_0, \rho, \eta) \subset \mathcal{C}W$ for some $\rho < \rho_0(\epsilon_0)$, then also $D(X_0, \epsilon_0, \rho/2, \eta) \subset \mathcal{C}W$ for some η'.

The consequences of Lemma 2 are:

(a) if $X_0 \in F$ is a positive density point for $\mathcal{C}W$ then any $Y_0 \in F$ near X_0 is of positive density for $\mathcal{C}W$, and, hence,

(b) in a neighborhood of X_0, $v_{ii}(Y) > - Cd(Y, F)^{\epsilon}$ for some $\epsilon > 0$; hence

(c) if $X_1 \in \overline{W}$ and $X_1 = X_0 + \lambda\eta$, where $X_0 \in F$, there is a curve $X = Q(\rho)$ $(2\lambda < \rho < \rho_0)$ in W along which $|Q(\rho) - X| = \rho$ and $\alpha(Q(\rho) - X_0, \eta) = \rho^{\epsilon'}$, for some $\epsilon' > 0$. This allows us to show that if X_0 is a point of density for $\mathcal{C}W$, then in a neighborhood of X_0, F is the graph of a Lipschitz function. An argument presented by the author [2] then gives the C^1 character of F and the C^2 character of v.

In the parabolic case, we begin with the variational inequality presentation of the Stefan problem of Duvaut [3] and Friedman and Kinderlehrer [4].

For simplicity we will assume a smooth initial domain Ω, which contains an initial open, smooth (C^2) subset I_0 with $\overline{I}_0 \subset \Omega$. ($I_0$ is the initial location of the ice.) $\Omega\backslash\overline{I}_0$ is connected and the temperature θ is prescribed on $\partial\Omega \times (0, T)$ and $(\Omega\backslash I_0) \times \{0\}$. We assume the boundary data for θ to be smooth (as in [4]), nonnegative, and not identically zero.

The variational solution [4] is given by a nonnegative function $v \in C_X^{1,1}(\Omega \times [\epsilon, T])$, $C_T^{0,1}(\Omega \times [0, T])$. The temperature θ is given by the bounded function v_T, and the ice is represented by the set $I = \{(X, t): v(X, t) = 0\}$. Since v_t is known to be nonnegative (a.e.), I can be represented by $\{(X, t): t < s(X)\}$, and although it is not explicit in [4], it follows easily from the approximating problems constructed there that (a) $\Delta v_t - v_{tt} = 0$ on $[\Omega \times (0, T)]\backslash I$, and (b) $\Delta v_t - v_{tt} \geqslant 0$ on $\Omega \times (0, T)$. Under these circumstances we can prove

THEOREM 2. (a) *If $(X_0, t_0) \in \partial I$ is a (spatial) density point for I, then there is a neighborhood $\{(X, t): |X - X_0| < \epsilon_0, |t - t_0| < \delta_0\}$, in which $F = \partial I$ is the graph of a function g; that is, in suitable coordinates $F = \{X_n = g(X_1, \ldots, X_{n-1}, t)\}$ where g is C^1 in all its variables and all the second derivatives v_{ij}, v_{ti}, v_{tt} are continuous up to F. In particular, v_t is (in this neighborhood) a classical solution of the Stefan problem.*

(b) *$s(X)$ is a Lipschitz function in any compact subset of I_0, and hence, for $n = 2$, v_t converges uniformly to zero on $\partial I \cap \{(X, t): \epsilon < t < T\}$.*

The proof of (a) is an adaptation of the techniques employed in the elliptic case. The regularity of v_{it}, v_{tt} and of g in time requires an extra argument. About (b), it is possible to prove that

$$v_t(X, T) > C \inf_{(Y,s)\in I, s<t} |X - Y| + (t - s)^{1/2}.$$

This implies that $s(x)$ is Lipschitz and therefore any point in ∂I is, for $n = 2$, regular for the exterior Dirichlet problem. (See the appendix by A. N. Milgram to the book *Partial differential equations*, by L. Bers, F. John and M. Schechter.)

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