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DISCONNECTED SOLUTIONS

BY W. F. LUCAS[1]

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1. **Introduction.** In the book, *Theory of games and economic behavior* (1944), J. von Neumann and O. Morgenstern introduced a theory of solutions (or stable sets) for multi-person cooperative games in characteristic function form. A longstanding conjecture has been that the *union* of all solutions of any particular game is a connected set. (E.g., see [3].) This announcement describes a twelve-person game for which the conjecture fails. The essential definitions for an n-person game will be reviewed briefly before the counterexample is presented. A sketch of the proof is presented here, and the details will appear elsewhere.

2. **The model.** An *n-person game* is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ is the set of *players* and v is a *characteristic function* on 2^N, i.e., v assigns the real number $v(S)$ to each subset S of N and $v(\emptyset) = 0$. The set of *imputations* is

$$A = \left\{x: \sum_{i \in N} x_i = v(N) \text{ and } x_i \geqslant v(\{i\}) \text{ for all } i \in N\right\}$$

where $x = (x_1, x_2, \ldots, x_n)$ is a vector with real components. For any $S \subset N$, let $x(S) = \Sigma_{i \in S} x_i$. For any $X \subset A$ and nonempty $S \subset N$, define $\mathrm{Dom}_S X$ to be the set of all $x \in A$ such that there exists a $y \in X$ with $y_i > x_i$ for all $i \in S$ and with $y(S) \leqslant v(S)$. Let $\mathrm{Dom}\, X = \bigcup_{\emptyset \neq S \subset N} \mathrm{Dom}_S X$. A subset V of A is a *solution* if $V \cap \mathrm{Dom}\, V = \emptyset$ and $V \cup \mathrm{Dom}\, V = A$. The *core* of a game is

$$C = \{x \in A: x(S) \geqslant v(S) \text{ for all nonempty } S \subset N\}.$$

For any solution V, $C \subset V$ and $V \cap \mathrm{Dom}\, C = \emptyset$.

A characteristic function v is *superadditive* if $v(S \cup T) \geqslant v(S) + v(T)$ whenever $S \cap T = \emptyset$. The game below does not have a superadditive v as is assumed in the classical theory, but it is equivalent solutionwise to a game with a superadditive v. (See [1, p. 68].)

3. **Example.** The 13 vital coalitions for our example consist of $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and elements from three classes:

$\mathcal{B} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$,

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$\mathcal{S} = \{\{1, 3, 6, 7, 9, 11\}, \{1, 4, 5, 7, 9, 11\}, \{2, 3, 5, 7, 9, 11\}\}$,
$\mathcal{T} = \{\{1, 3, 8\}, \{1, 5, 10\}, \{3, 5, 12\}\}$.

And v is given by: $v(N) = 6$, $v(S) = 1$ for all $S \in \mathcal{B}$, $v(S) = 4$ for all $S \in \mathcal{S}$, $v(S) = 1$ for all $S \in \mathcal{T}$, and $v(S) = 0$ for all other $S \subset N$. For this game $A = \{x\colon x(N) = 6 \text{ and } x_i \geqslant 0 \text{ for all } i \in N\}$. Consider also the six-dimensional hypercube

$$B = \{x \in A\colon x(S) = 1 \text{ for all } S \in \mathcal{B}\}.$$

The core C is the intersection of $C(\mathcal{S})$ and $C(\mathcal{T})$ where

$$C(\mathcal{S}) = \{x \in B\colon x(S) \geqslant 4 \text{ for all } S \in \mathcal{S}\},$$
$$C(\mathcal{T}) = \{x \in B\colon x(S) \geqslant 1 \text{ for all } S \in \mathcal{T}\}.$$

C is a proper superset of the convex hull of the six vertices of B which have $x_i = 1$ for i equal to five of the six odd indices 1, 3, 5, 7, 9 and 11, and $x_{i+1} = 1$ when i is the remaining odd numbered player. Let $\text{Dom}_{\mathcal{B}} X = \bigcup_{S \in \mathcal{B}} \text{Dom}_S X$. Note that $\text{Dom}_{\mathcal{B}} C \supset A - B$, and hence any solution V for our game is a subset of B.

4. **Outline of proof.** First, note that any component of an $x \in B$ has a maximum value of $x_i = 1$. Consequently, the following three sets are contained in any solution V, i.e., they are subsets of $\bigcap V$:

$E = \{x \in B\colon x_i = x_j = 1 \text{ for } i \neq j \text{ and } \{i, j\} \subset \{1, 3, 5\}\}$,
$F = \{x \in C(\mathcal{T})\colon x_p = 1 \text{ for } p = 7, 9 \text{ or } 11\}$,
$P = \{(0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1)\}$.

Next, we can show that $\bigcup V$ must be a disconnected set. Let $G = \{x \in B\colon x(\{7, 9, 11\}) \leqslant 1\}$, $G^0 = \{x \in B\colon x(\{7, 9, 11\}) < 1\}$, and $P' = \{x \in G\colon x_2 = x_4 = x_6 = 1\}$. Throughout this section the indices i, j and k represent some ordering of the distinct indices 1, 3 and 5. The subset H of E consisting of the three triangular regions

$$H_i = \{x \in G\colon x_{i+1} = x_j = x_k = 1; x_7 + x_9 + x_{11} = 1\}$$

is in $\bigcap V$ and $\text{Dom}_{\mathcal{S}} N \supset G^0 - (E \cup P')$. The subset J of F consisting of the three triangular regions

$J_1 = \{x \in F\colon x_1 = x_7 = x_9 = 1, x_3 + x_5 + x_{12} = 1\}$,
$J_3 = \{x \in F\colon x_3 = x_7 = x_{11} = 1, x_1 + x_5 + x_{10} = 1\}$,
$J_5 = \{x \in F\colon x_5 = x_9 = x_{11} = 1, x_1 + x_3 + x_8 = 1\}$

is also in $\bigcap V$ and $\text{Dom}_{\mathcal{T}} J \supset B - C(\mathcal{T}) \supset P' - P$. So any $x \in \bigcup V - P$ either has $x \in E$ or $x \in B - G^0$, i.e., $x_i = x_j = 1$ or $x(\{7, 9, 11\}) \geqslant 1$. Such x are clearly disconnected from the singleton $P \subset \bigcap V$.

Finally, it is necessary to demonstrate that this game does possess at least one solution. $V' = C \cup E \cup F \cup P$ is in any solution V, and V' can be enlarged to a solution in two steps. First, include the set of imputations L in $C(\mathcal{T}) -$

$(V' \cup \text{Dom } V')$ which is simultaneously maximal with respect to all three of the relations "Dom_S" for $S \in \mathcal{S}$. Clearly $L \subset \bigcap V$. Next, pick a particular $S^i = \{i + 1, j, k, 7, 9, 11\} \in \mathcal{S}$ and then add in those elements L^i in $C(T) - (V' \cup L \cup \text{Dom}(V' \cup L))$ which are maximal with respect to the relation "Dom_{S^i}" and are at the same time symmetrical in the sense that $x_j = x_k$. It requires some detail to describe the sets L and L^i explicitly, and to verify that the resulting sets $V^i = V' \cup L \cup L^i$ are solutions for our example. These will appear elsewhere

5. **Remarks.** At one time it was apparently believed that proving the union of all solutions connected could be a major step in showing that every game has a solution. It is now known [2] that a solution need not exist for every game. On the other hand, it is possible that results on disconnecting $\bigcup V$ might be useful in the resolution of important open questions about whether solutions always exist for games with full-dimensional cores, with empty cores, or which are constant-sum.

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SCHOOL OF OPERATIONS RESEARCH AND CENTER FOR APPLIED MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853