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GLOBAL BIFURCATION THEOREMS FOR NONLINEARLY PERTURBED OPERATOR EQUATIONS

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1. **Introduction.** The author [2], [3], and [4] has previously studied the equation

$$Lu = \lambda u + H(\lambda, u) \tag{1}$$

in a real Banach space B where L is linear and H is compact and $o(\|u\|)$ is uniformly on bounded λ intervals. In that setting, if λ_0 is an isolated normal eigenvalue of L having odd algebraic multiplicity, then $(\lambda_0, 0) \in R \times B$ is a bifurcation point for (1). Moreover, a continuous branch of solutions emanates from each of these points and obeys a threefold alternative.

This paper combines methods of the author and Stuart [7] to show that similar results hold if $H(\lambda, u)$ is merely continuous and $o(\|u\|)$ uniformly on bounded λ intervals.

2. **Preliminaries.** In this paper all work is a real Banach space B. L denotes a linear operator densely defined in B, and H represents a continuous operator that is $o(\|u\|)$ near $u = 0$ uniformly on bounded λ intervals. Define the essential spectrum of L as the members of the spectrum of L which are not isolated normal eigenvalues of L and denote this set by $e(L)$.

We consider a normal eigenvalue λ_0 of L. Let

$$\alpha_{\lambda_0} = \sup\{\gamma \mid \gamma \in e(L), \gamma < \lambda_0\} \quad \text{and} \quad \beta_{\lambda_0} = \inf\{\gamma \mid \gamma \in e(L), \gamma > \lambda_0\}$$

respectively if the corresponding sup or inf exists. Otherwise, set $\alpha_{\lambda_0} = -\infty$ and/or $\beta_{\lambda_0} = +\infty$. Assume for now that α_{λ_0} and β_{λ_0} are both finite. For $\epsilon > 0$, the only members of the spectrum of L in $(\alpha_{\lambda_0} + \epsilon, \beta_{\lambda_0} - \epsilon)$ are normal eigenvalues of L. If P_ϵ denotes the projector onto the direct sum of the eigenspaces of these eigenvalues and $Q_\epsilon = I - P_\epsilon$, then it has been shown [2], [3] and [4] that

$$u = \frac{(L - \mu_0)P_\epsilon u}{\lambda - \mu_0} + \left((L - \lambda)^{-1}Q_\epsilon - \frac{P_\epsilon}{\lambda - \mu_0}\right) H(\lambda, u) \tag{2}$$

is equivalent to (1) for λ in $[\alpha_{\lambda_0} + \epsilon, \beta_{\lambda_0} - \epsilon]$ and μ_0 any member of the resolvent of L not lying in $(\alpha_{\lambda_0}, \beta_{\lambda_0})$ ($(L - \lambda)^{-1}$ is defined on $Q_\epsilon B$).

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Define

$$M(\lambda, \gamma) = \sup \left\{ \frac{\|H(\lambda, u)\|}{\|u\|} \,\middle|\, 0 < \|u\| \leqslant \gamma \right\}.$$

Clearly $\lim_{\gamma \to 0} M(\lambda, \gamma) = 0$.

3. **Results.** Consider equation (1) in a real Banach space B, with L linear and H continuous, $o(\|u\|)$ uniformly on bounded intervals.

THEOREM I. *Let λ_0 be an isolated eigenvalue of L having odd algebraic multiplicity. Then $(\lambda_0, 0)$ is a bifurcation point for* (2) *and emanating from that point there is a maximal continuous branch of solutions C_{λ_0} that must obey the following alternative*:

(a) C_{λ_0} *is unbounded, or*

(b) C_{λ_0} *is bounded and for each $\delta > 0$, C_{λ_0} meets the surface*

$$S_\epsilon = \{(\lambda, u) \mid \|(L - \lambda)^{-1} Q_\epsilon\| M(\lambda, \gamma) = 1,$$

$$\|u\| = \gamma, \quad \alpha_{\lambda_0} + \epsilon < \lambda < \beta_{\lambda_0} - \epsilon\} \quad \textit{for some } \epsilon, 0 < \epsilon < \delta,$$

(c) C_{λ_0} *is bounded, $\overline{C_{\lambda_0}}$ does not meet S_ϵ for all $\epsilon \in (0, \delta)$ for some $\delta > 0$, and $\overline{C_{\lambda_0}} \cap \{0 \times B\} = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$, each a distinct normal eigenvalue of L, and the sum of their algebraic multiplicities being even.*

PROOF. Assume for now that α_{λ_0} and β_{λ_0} are finite, and that C_{λ_0} does not obey any of the three alternatives. Then C_{λ_0} consists of pairs (λ, u) with $\alpha_{\lambda_0} + 2\epsilon < \lambda < \beta_{\lambda_0} - 2\epsilon$ for some $\epsilon > 0$. Using this ϵ, one sees that the operators on the right side of (2) are a k-set contraction with

$$k = \|(L - \lambda)^{-1} Q_\epsilon\| M(\lambda, \gamma) \quad \text{for each } \lambda \in (\alpha_{\lambda_0} + \epsilon, \beta_{\lambda_0} - \epsilon),$$

and u with $\|u\| \leqslant \gamma$. We may further assume that C_{λ_0} does not meet S_ϵ. If S_ϵ is viewed in $\mathbf{R} \times \|B\|$, we see that $k < 1$ below S_ϵ. Techniques developed in [2] and [6] which employ degree theory for k-set contractions [5] lead to a contradiction.

In the case that α_{λ_0} or β_{λ_0} are infinite, the result is obtained using an iterative procedure.

In the case that L is selfadjoint, the results are simpler. Let $d(\lambda) = \min(\lambda - \alpha_{\lambda_0}, \beta_{\lambda_0} - \lambda)$.

COROLLARY 1. *If the assumptions of Theorem* I *hold and if, moreover, L is selfadjoint, then alternatives* (b) *and* (c) *are equivalent to*

(b′) C_{λ_0} *is bounded and $\overline{C_{\lambda_0}}$ meets*

$$S = \left\{ (\lambda, u) \,\middle|\, \frac{M(\lambda, \gamma)}{d(\lambda)} = 1, \|u\| = \gamma, \alpha_{\lambda_0} < \lambda < \beta_{\lambda_0} \right\}$$

(c′) C_{λ_0} *is bounded*, $\overline{C_{\lambda_0}}$ *does not meet* S, *and* $C_{\lambda_0} \cap \{0 \times B\} = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$, *each a distinct normal eigenvalue of* L, *and the sum of their algebraic multiplicities being even.*

PROOF. With L being selfadjoint and $\alpha_{\lambda_0} + \epsilon < \lambda < \beta_{\lambda_0} - \epsilon$,

$$\|(L - \lambda)^{-1} Q_\epsilon\| = \|(L - \lambda)^{-1} | Q_\epsilon B\| \|Q_\epsilon\| = \|(L - \lambda)^{-1} | Q_\epsilon B\|.$$

As ϵ goes to 0, $\|(L - \lambda)^{-1} | Q_\epsilon B\|$ approaches $\|(L - \lambda)^{-1} Q_0 B\| = 1/d(\lambda)$ and S_ϵ approaches S.

REFERENCES

1. M. A. Krasnosel′skiĭ, *Topological methods in the theory of nonlinear integral equations*, GITTL, Moscow, 1956; English transl., Macmillan, New York, 1964. MR **20** # 3464; **28** # 2414.

2. J. A. MacBain, *Local and global bifurcation from normal eigenvalues*, Ph. D. Thesis, Purdue Univ., 1974.

3. ———, *Global bifurcation theorems for noncompact operators*, Bull. Amer. Math. Soc. **80** (1974), 1005–1009. MR **50** # 14403.

4. ———, *Local and global bifurcation from normal eigenvalues*, Pacific J. Math. **63** (1976).

5. R. D. Nussbaum, *The fixed point index and fixed point theorems for k-set contractions*, Ph.D. Thesis, Chicago, Ill., 1969.

6. P. H. Rabinowitz, *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain Consortium Sympos. on Nonlinear Eigenvalue Problems (Santa Fe, N. M., 1971), Rocky Mountain J. Math. **3** (1973), 161–202. MR **47** # 9383.

7. C. A. Stuart, *Some bifurcation theory for k-set contractions*, Proc. London Math. Soc. (3) **27** (1973), 531–550.

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