BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, July 1976

FORM OF SOLUTIONS TO THE p-ADIC EQUATION $y' = 0$

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Communicated by John W. Wrench, October 13, 1975

The solutions of $y' = 0$ over the real and/or complex numbers have long been known to be the constant functions. Here we shall present a form for any function, z, that maps a suitable subset of $\mathbf{Q}_p$, the complete field of p-adic numbers, into $\mathbf{Q}_p$ (where p is a positive prime) and is differentiable with derivative zero everywhere. We shall also discuss the image-set of such a function.

This problem has been of interest ever since J. Dieudonné gave an example [1, p. 90], [2, p. 19], [4, pp. 35, 39] of a function, z_0, that maps $\mathbf{Z}_p$ homeomorphically onto its image-set and yet has a zero derivative everywhere. M. van der Put has studied integration of p-adic valued functions [3] using the set of solutions to $y' = 0$ without determining this set [2, p. 20].

Let $\mathbf{N}$ be the set of nonnegative integers, let $\mathbf{R}$ be the set of real numbers, and let $\mathbf{R}_{>b}$ be the set of all real numbers greater than the real number b. Let $C = \{0, 1, 2, 3, \ldots, p-1\}$ and let $\mathbf{Z}_p$ be the set of p-adic integers. Every p-adic integer has a canonical form $\Sigma\{a_j p^j : j \in \mathbf{N}\}$, where each a_j is an element of C. z_0, the function of Dieudonné, is given by

$$z_0\left(\sum\{a_j p^j : j \in \mathbf{N}\}\right) = \sum\{a_j p^{2j} : j \in \mathbf{N}\}.$$

First we let f be a function mapping a subspace of $\mathbf{Z}_p$ into $\mathbf{Z}_p$. It is easily shown that f is uniformly continuous (on its domain) iff

$$(\exists l: \mathbf{N} \longrightarrow \mathbf{N})\,(\forall n \in \mathbf{N})\,(\exists g_n: C^{l(n)} \longrightarrow \mathbf{Z}_p)\quad(\forall a \in C^{\mathbf{N}})$$
$$\sum\{a_j p^j : j \in \mathbf{N}\} \in \mathrm{Dom}(f) \Rightarrow$$
$$f\left(\sum\{a_j p^j : j \in \mathbf{N}\}\right) = \sum\{g_n(a_0, a_1, a_2, a_3, \ldots, a_{l(n)-1})p^n : n \in \mathbf{N}\}.$$

Now the concept of uniform differentiability is introduced; it bears the same relationship to differentiability that uniform continuity has to continuity. Formally, f is uniformly differentiable on D' iff D' is a subset of $\mathrm{Dom}(f)$ that contains no isolated points of $\mathrm{Dom}(f)$ and

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AMS (MOS) subject classifications (1970). Primary 34G05, 12B99; Secondary 28-00, 43A75, 43A85.

Key words and phrases. Solutions of $y' = 0$, p-adic solutions of $y' = 0$, kernel of differentiation operator, Egoroff's theorem, uniform continuity, uniform differentiability.

$$(\forall \in \epsilon\, \mathbf{R}_{>0})\,(\exists\ \delta \in \mathbf{R}_{>0})\,(\forall\, x \in D')\,(\exists\ w \in \mathbf{Q}_p)\ \ (\forall\, y \in \mathrm{Dom}(f))$$

$$0 < |y - x| < \delta \implies |(f(y) - f(x))/(y - x) - w| < \epsilon.$$

Second, we show that a function f is uniformly differentiable with derivative zero everywhere iff it has the form (stated above) for uniformly continuous functions and $n - l(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ (l may have to be chosen so that $l(n)$ is the least value "that works"). Furthermore, the image-set of such a function has Jordan content zero, consequently it is nowhere dense.

However, there exist functions that map $\mathbf{Z}_p$ into $\mathbf{Z}_p$ and are differentiable with derivative zero everywhere, yet are not uniformly differentiable. One such function can be exhibited by putting $C' = C\backslash\{0\}$ and defining z_1 on $\mathbf{Z}_p$ by $z_1(x) = p^{2k}$ if, for some $k \in \mathbf{N}$, $x \in C'p^k + C'p^{4k+4} + \mathbf{Z}_p p^{4k+5}$ and $z_1(x) = 0$ otherwise. The least function l "that works" for z_1 is

$$l(n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2n + 4 & \text{if } n \text{ is even.} \end{cases}$$

Third, a differentiable function with a continuous derivative on a measurable (with respect to a measure that is nonnegative, regular, and finite on compact sets) domain can have this domain expressed, except for a set of measure 0, as the union of a nondecreasing sequence of compact sets, on each of which the given function is uniformly differentiable (Shades of Egoroff's Theorem). As something of a converse, if f is a function and D is a nondecreasing sequence of sets on each of which f is uniformly differentiable, then f is differentiable on the union of D.

Fourth and finally, the second and third results are combined to give a form for the solutions of $y' = 0$ over the p-adics.

In the course of the paper [**5**], uniform differentiability is related to other concepts of mathematics of the same ilk, e.g., uniform differentiability and bounded derivative imply uniform continuity.

Details will appear in the paper [**5**], and generalizations in another paper.

Both [**5**] and the current paper overlap slightly with [**6**, pp. 90–91]. There another form is given for a uniformly differentiable function defined on $\mathbf{Z}_p$ with derivative zero everywhere that takes its values in a valued field containing $\mathbf{Q}_p$.

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