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ON THE NUMBER OF INVARIANT CLOSED GEODESICS

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It is an outstanding problem in riemannian geometry whether any compact riemannian manifold of dimension $n + 1 > 1$ has infinitely many closed geodesics. In this note we outline a proof of the following:

THEOREM. *Let M be a compact, 1-connected riemannian manifold and $A: M \longrightarrow M$ an isometry of finite order. Then A has infinitely many closed invariant geodesics if the sequence of Betti numbers for the space of maps $\sigma: \mathbf{R} \longrightarrow M$ with $\sigma(t + 1) = A(\sigma(t))$ is unbounded.*

This is a generalization of a well-known theorem on closed geodesics ($A = 1_M$) by Gromoll and Meyer [2]. Observe that the assumption on the Betti numbers in our theorem is essential (A = rotation on S^2). Note also that the isometries of finite order are dense in the isometry group.

OUTLINE OF PROOF. Let $\Lambda(M, A)$ be the complete, riemannian Hilbert manifold of all absolutely continuous maps $\sigma: \mathbf{R} \longrightarrow M$ with $\dot{\sigma}: \mathbf{R} \longrightarrow TM$ locally square integrable and $\sigma(t + 1) = A(\sigma(t))$ [4]. The critical points for the energy integral $E^A: \Lambda(M, A) \longrightarrow \mathbf{R}$ correspond to A-invariant geodesics, and E^A satisfies condition (C) of Palais and Smale [4]. The fixed point set of A, Fix(A) corresponds to the critical points with E^A-value zero, and it consists of finitely many nondegenerate critical submanifolds of $\Lambda(M, A)$. The contribution of Fix(A) to the homology of $\Lambda(M, A)$ is therefore at most finite dimensional.

The $\mathbf{R}$-action on $\Lambda(M, A)$ induced by translation of the parameter reduces to an $S^1 = \mathbf{R}/s \cdot \mathbf{Z}$-action, when A has order $s \in \mathbf{Z}^+$. If γ is a nontrivial closed A-invariant geodesic, it is represented by a critical point $c \in \Lambda(M, A)$ whose fundamental period is s/m for some integer $m \leqslant s$. Let $s/m = s_0/m_0$, where s_0 and m_0 are relatively prime positive integers, and choose integers n_0 and k_0 such that $m_0 n_0 = 1 + s_0 k_0$. Define $c^u: \mathbf{R} \longrightarrow M$ for any $u \in \mathbf{R}$ by $c^u(t) = c(u \cdot t)$ and put $\bar{c} = c^{1/m_0}$. Then $\bar{c}$ is a critical point for $E^{A^{n_0}}$ with fundamental period s_0 and $\bar{c} \subset \text{Fix}(A^{s_0})$. For any integers m and r with $ms_0 + rm_0 \neq 0$, $\bar{c}^{ms_0 + rm_0}$ is a critical point for E^{A^r} and $S^1 \cdot c^{ms_0 + m_0}$, $m \in \mathbf{Z}^+ \cup \{0\}$ are all the critical orbits in $\Lambda(M, A)$ "generated" by γ. In analogy to Bott [1] we find formulas for the indices and nullities of the critical orbits $S^1 \cdot \bar{c}^{ms_0 + rm_0}$ in $\Lambda(M, A^r)$ from which we derive:

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LEMMA 1. *For each integer* $0 \leqslant 1 < s/s_0$, *either* $\lambda(\bar{c}^{ms_0+m_0}, A) = 0$ *for all* $m \in D_1 := \{m \in \mathbf{Z}^+ \cup \{0\} \mid mn_0 + k_0 \equiv 1 \bmod s/s_0\}$ *or there exist* $\epsilon_1, a_1 \in \mathbf{R}^+$ *such that*

$$\lambda(\bar{c}^{m_1s_0+m_0}, A) - \lambda(\bar{c}^{m_2s_0+m_0}, A) \geqslant (m_1 - m_2)\epsilon_1 - a_1$$

for all $m_1, m_2 \in D_1$ *with* $m_1 \geqslant m_2$.

LEMMA 2. *For each integer* $0 \leqslant 1 < s/s_0$, *there exist* $k_1, \ldots, k_q \in \mathbf{Z}^+$ *and* $\{m_j^i\} \subset \mathbf{Z}^+$, $j = 1, \ldots, q$, $i > 0$, *such that the numbers* $\{m_j^i k_j\}$ *are mutually distinct* $\{m_j^i k_j\} = \{ms_0 + m_0 \mid m \in D_1\}$ *and*

$$\nu(\bar{c}^{m_j^i k_j}, A) = \nu(\bar{c}^{m_j^i k_j}, A \mid \mathrm{Fix}(A^{s_0 s_j^i})) = \nu(\bar{c}^{k_j}, A^r \mid \mathrm{Fix}(A^{s_0 s_j^i})),$$

where s_j^i *is maximal with the properties* $(m_j^i, s_j^i) = 1$ *and* $s_j^i \mid s/s_0$, *and where* $r \in \mathbf{Z}$ *satisfies* $rm_j^i \equiv 1 \bmod s_0 s_j^i$.

To each isolated orbit $S^1 \cdot c^{ms_0+m_0}$ there is associated a local homological invariant $H(S^1 \cdot \bar{c}^{ms_0+m_0}, E^A)$ which by the "generalized" Morse inequalities gives an upper bound for the contribution of $S^1 \cdot c^{ms_0+m_0}$ to the homology of $\Lambda(M, A)$ [2], [3]. The local invariant $H(\bar{c}^{ms_0+m_0}, E^A)$ is completely determined by the index $\lambda(\bar{c}^{ms_0+m_0}, A)$ and a characteristic invariant $H^0(\bar{c}^{ms_0+m_0}, E^A)$, which in turn is determined by the degenerate part of E^A [3].

Under the assumption that there are only finitely many closed A-invariant geodesics on M it follows from Lemmas 1 and 2, that there are only finitely many different characteristic invariants among $\{H^0(\bar{c}^{ms_0+m_0}, E^A) \mid m \in \mathbf{Z}^+ \cup \{0\}\}$. Furthermore, for large k the number of orbits with

$$\dim H_k(S^1 \cdot \bar{c}^{ms_0+m_0}, E^A) \neq 0$$

is uniformly bounded. Using these properties we conclude that the sequence of Betti numbers for $\Lambda(M, A)$ is bounded. Full details will appear elsewhere.

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