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# MAYER-VIETORIS SEQUENCES FOR COMPLEXES OF DIFFERENTIAL OPERATORS

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This is an announcement of some of the results in [1].

1. **Preliminaries.** Let $X$ be a smooth manifold, $E^i$, $i = 0, 1, \ldots$, smooth vector bundles, and $\Omega \subset X$ open. Let $\mathcal{E}^i(\Omega) = C^\infty(\Omega, E^i)$. We consider complexes of linear differential operators with locally constant orders

$$\mathcal{E}(\Omega)\colon\ \mathcal{E}^0(\Omega) \xrightarrow{D^0} \mathcal{E}^1(\Omega) \xrightarrow{D^1} \cdots .$$

The cohomology of $\mathcal{E}(\Omega)$ is $H^i(\Omega) = \ker D^i/\mathrm{im}\, D^{i-1}$. Let $S \subset \Omega$ be a smooth hypersurface dividing $\Omega$ into two parts: $\Omega - S = \mathring{\Omega}^+ \cup \mathring{\Omega}^-$; $\mathring{\Omega}^+ \cap \mathring{\Omega}^- = \varnothing$; and $S \cup \mathring{\Omega}^\pm = \Omega^\pm$. Let $\mathcal{E}^i(\Omega^\pm)$ be the sections over $\Omega^\pm$ smooth up to $S$. We obtain

$$\mathcal{E}(\Omega^\pm)\colon\ \mathcal{E}^0(\Omega^\pm) \xrightarrow{D^0} \mathcal{E}^1(\Omega^\pm) \xrightarrow{D^1} \cdots .$$

A section $u \in \mathcal{E}^i(\Omega)$ has zero Cauchy data on $S$ if $D^i\widetilde{u} = \widetilde{f}$ is valid on $\Omega$ in the sense of distributions where $\widetilde{u} = u$ on $\Omega^+$ and $= 0$ on $\Omega - \Omega^+$, and $\widetilde{f} = D^iu$ on $\Omega^+$ and $= 0$ on $\Omega - \Omega^+$; and similarly with $\Omega^+$ replaced by $\Omega^-$. The space of such sections is $\mathcal{I}(\Omega, S)$, and $\mathcal{I}(\Omega^\pm, S) = \mathcal{I}(\Omega, S)|_{\Omega^\pm}$. We obtain complexes

$$\mathcal{I}(\Omega, S)\colon\ \mathcal{I}^0(\Omega, S) \xrightarrow{D^0} \mathcal{I}^1(\Omega, S) \xrightarrow{D^1} \cdots ,$$

and

$$\mathcal{I}(\Omega^\pm, S)\colon\ \mathcal{I}^0(\Omega^\pm, S) \xrightarrow{D^0} \mathcal{I}^1(\Omega^\pm, S) \xrightarrow{D^1} \cdots ,$$

with cohomologies $H^i(\Omega, \mathcal{I})$ and $H^i(\Omega^\pm, \mathcal{I})$, respectively.

The tangential complex is the quotient complex $0 \longrightarrow \mathcal{I}(\Omega, S) \longrightarrow \mathcal{E}(\Omega) \longrightarrow \mathcal{C}(S) \longrightarrow 0$. An element of $\mathcal{C}^i(S)$ is Cauchy data for $D^i$, the induced operator is $D^i_S$, and the cohomology is $H^i(S)$.

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Let $F(\Omega, S)$ (resp. $F(\Omega^{\pm}, S)$ be the space of sections of $E(\Omega)$ (resp. $E(\Omega^{\pm})$) which vanish to infinite order on $S$. The quotient $E/F(S)$ may be thought of as sequences of sections over $S$ representing the normal derivatives of all orders of sections in $E(\Omega)$. The diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ & & 0 & \longrightarrow & C(S) & \longrightarrow & C/F & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F(\Omega) & \longrightarrow & E(\Omega) & \longrightarrow & E/F & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & F(\Omega) & \longrightarrow & I(\Omega) & \longrightarrow & I/F & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{1}$$

commutes and has exact rows and columns.

The cohomology of

$$0 \longrightarrow (I/F)^0 \xrightarrow{D^0} (I/F)^1 \xrightarrow{D^1} \cdots, \tag{2}$$

denoted by $H^i(I/F)$, is the Cauchy-Kowalewski cohomology of $E(\Omega)$ on $S$.

DEFINITION. *$S$ is formally noncharacteristic* for $E$ if $H^i(I/F) = 0$, all $i \geqslant 0$.

## 2. The Mayer-Vietoris sequence in cohomology.

THEOREM 1. *If $S$ is formally noncharacteristic for $E$, then*

$$0 \longrightarrow H^0(\Omega) \longrightarrow H^0(\Omega^+) \oplus H^0(\Omega^-) \longrightarrow H^0(S) \longrightarrow H^1(\Omega) \longrightarrow \cdots \tag{3}$$

*is an exact sequence.*

PROOF. By the Whitney extension theorem

$$0 \longrightarrow E(\Omega) \longrightarrow E(\Omega^+) \oplus E(\Omega^-) \longrightarrow E/F \longrightarrow 0$$

is an exact sequence, where the first map is restriction and the second is the jump in normal derivatives. The long exact sequence of this gives (3) with $H^i(E/F)$ instead of $H^i(S)$. But the long exact sequence of the last column of (1), with $H^i(I/F) = 0$, gives $H^i(E/F) \cong H^i(S)$.

The proof of the following is in the same spirit.

THEOREM 2. *If $S$ is formally noncharacteristic for $E$, then*

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0(\Omega^\mp, \mathcal{I}) & \longrightarrow & H^0(\Omega) & \longrightarrow & H^0(\Omega^\pm) & \longrightarrow & H^1(\Omega^\mp, \mathcal{I}) & \longrightarrow \cdots \\
& \downarrow \| & & \downarrow & & \downarrow & & \downarrow \| & \\
0 \longrightarrow & H^0(\Omega^\mp, \mathcal{I}) & \longrightarrow & H^0(\Omega^\mp) & \longrightarrow & H^0(S) & \longrightarrow & H^1(\Omega^\mp, \mathcal{I}) & \longrightarrow \cdots
\end{array}$$

*commutes, and has exact rows.*

DEFINITION. A cotangent vector $\xi \in T^*_x(X)$ is *noncharacteristic* for $\mathcal{E}$ if the principal symbol complex

$$0 \longrightarrow E^0_x \xrightarrow{\sigma_\xi(D^0)} E^1_x \xrightarrow{\sigma_\xi(D^1)} \cdots$$

is exact. The surface $S$ is noncharacteristic if the normal cotangent vector field is noncharacteristic at each point.

THEOREM 3. *If $S$ is noncharacteristic for $\mathcal{E}$, then $S$ is formally noncharacteristic for $\mathcal{E}$.*

This is a formal version of the Cauchy-Kowalewski theorem for complexes of operators.

If $\mathcal{E}(\Omega)$ and $\mathcal{E}(\Omega^\pm)$ are given the topology of uniform convergence of all derivatives on compact sets, and if subspaces and quotients are given the induced and quotient topologies, the maps in Theorems 1 and 2 are continuous.

REMARK. We may consider the spaces of sections with support in any regular paracompactifying family of supports, in particular compact supports, topologized as the strict inductive limits of Fréchet-Schwartz spaces, and all the above results still hold.

3. **The Mayer-Vietoris sequence in homology.** Consider the topological duals of the above complexes.

Denote the homology of

$$0 \longleftarrow (\mathcal{E}^0(\Omega))' \xleftarrow{(D^0)'} (\mathcal{E}^1(\Omega))' \xleftarrow{(D^1)'} \cdots$$

by $H_i(\Omega)$, and similarly for the other complexes.

THEOREM 4. *If $S$ is formally noncharacteristic,*

$$0 \longleftarrow H_0(\Omega) \longleftarrow H_0(\Omega^+) \oplus H_0(\Omega^-) \longleftarrow H_0(S) \longleftarrow H_1(\Omega) \longleftarrow \cdots$$

*is exact, and*

$$\begin{array}{ccccccccc}
0 \longleftarrow & H_0(\Omega^\mp, \mathcal{I}) & \longleftarrow & H_0(\Omega) & \longleftarrow & H_0(\Omega^\pm) & \longleftarrow & H_1(\Omega^\mp, \mathcal{I}) & \longleftarrow \cdots \\
& \uparrow \| & & \uparrow & & \uparrow & & \uparrow & \\
0 \longleftarrow & H_0(\Omega^\mp, \mathcal{I}) & \longleftarrow & H_0(\Omega^\mp) & \longleftarrow & H_0(S) & \longleftarrow & H_1(\Omega^\mp, \mathcal{I}) & \longleftarrow \cdots
\end{array}$$

*commutes and has exact rows.*

The open mapping theorem for Fréchet-Schwartz spaces shows that if $S$ is formally noncharacteristic, the dual of (2) is exact. The rest of the proof is as in Theorems 1 and 2.

The proof fails in the case of sections with supports in a regular paracompactifying family of supports. However, if $S$ is noncharacteristic, the proof of Theorem 3 can be adapted to show the dual of (2) is exact, and the theorem analogous to Theorem 4 for sections with restricted supports holds. Similar methods give the analogues of Theorems 1 and 2 for distributions. One also obtains duality theorems between homology and cohomology.

## REFERENCES

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