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ITERATED LOOP FUNCTORS AND THE HOMOLOGY OF THE STEENROD ALGEBRA

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Let A be the mod-2 Steenrod algebra. For any unstable A-module M the "unstable homology groups" $H^A_{s,k}(M) = \mathrm{Tor}^A_{s,k}(M)$ are defined by means of unstable projective resolutions of M [2]. We describe here a new approach to the problem of computing these groups.

Let $\mathcal{M}_A$ be the category whose objects are unstable A-modules and whose morphisms are degree preserving A-maps. For M in $\mathcal{M}_A$ and x in M_n we write, as is usual $\mathrm{Sq}_a x = \mathrm{Sq}^{n-a}x$. Let "suspension" S: $\mathcal{M}_A \longrightarrow \mathcal{M}_A$ be the functor that raises degree by 1. S has a left adjoint Ω: $\mathcal{M}_A \longrightarrow \mathcal{M}_A$ [2] given by $(\Omega M)_n = (\mathrm{coker}\, \mathrm{Sq}_0)_{n+1}$, with A-action induced by that on M. The left derived functors Ω_s $(s \geqslant 0)$ of Ω are defined in the usual way: given M in $\mathcal{M}_A$ one forms a projective resolution $\cdots \longrightarrow P_1(M) \longrightarrow P_0(M) \longrightarrow M \longrightarrow 0$. Then $\Omega_s M$ is the sth homology group of the complex $\cdots \longrightarrow \Omega P_1(M) \longrightarrow \Omega P_0(M) \longrightarrow 0$. The left derived functors of Ω are completely understood [1], [2], [3]. In fact,

$$\Omega_s M = 0 \quad \text{if } s > 1, \tag{1}$$

$$(\Omega_1 M)_{2n-1} = (\ker \mathrm{Sq}_0)_n \tag{2}$$

with A-action given by $\mathrm{Sq}_a \Omega_1 x = \Omega_1 \mathrm{Sq}_{(a+1)/2}$ for x in $\ker \mathrm{Sq}_0$.

Consider now the k-fold iterate Ω^k of Ω. We pose:

PROBLEM (*). Give a workable description of the left derived functors Ω^k_s of Ω^k, for all $s \geqslant 0$.

Our interest in these derived functors stems from the fact that their zero-dimensional components are the unstable homology groups of the Steenrod algebra:

THEOREM 1. *There is a natural isomorphism* $\mathrm{Tor}^A_{s,k}(M) = (\Omega^k_s M)_0$.

Our interest in Problem (*) is heightened by the fact that it appears to be solvable: there is a simple relation between the derived functors of Ω^k and those of Ω^{k-1}.

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THEOREM 2. *There is a natural long exact sequence of Z_2-modules*:

$$\cdots \longrightarrow \Omega_s^{k-1}M \xrightarrow{\mathrm{Sq}_0} \Omega_s^{k-1}M \longrightarrow \Omega_s^k M \longrightarrow \cdots \tag{3}$$
$$\longrightarrow \Omega_{s-1}^{k-1}M \xrightarrow{\mathrm{Sq}_0} \Omega_{s-1}^{k-1}M \longrightarrow \cdots$$

and consequently a short exact sequence in $\mathcal{M}_A$:

$$0 \longrightarrow \Omega\Omega_s^{k-1}M \longrightarrow \Omega_s^k M \longrightarrow \Omega_1\Omega_{s-1}^{k-1}M \longrightarrow 0. \tag{4}$$

This result seems to promise a quick inductive description of the functors Ω_s^k; however, computation of examples with small values of k and s show that the short exact sequence (4) is in general not split over $\mathcal{M}_A$!

Our main result (Theorem 3 below) is the construction for each unstable A-module M of a small chain complex $L^kM = \Sigma_s L_s^k M$ from which the derived functions $\Omega_s^k M$ can be computed: $H_s(L^kM) = \Omega_s^k M$. We seek, in particular, complexes that can be fit into a short exact sequence:

$$0 \longrightarrow L^{k-1}M \xrightarrow{\alpha} L^kM \xrightarrow{\beta} L^{k-1}M \longrightarrow 0 \tag{5}$$

for which the associated long exact sequence in homology is the same as (3). This consideration motivates us in our definition of the graded Z_2-module $L_s^kM = \Sigma_{n\geqslant 0}(L_s^kM)_n$: we set $(L_0^kM)_n = M_{n+k}$, and proceed inductively by setting $L_s^kM = \Sigma_{i=0}^{k-1}(L_{s-1}^iM)$, defining dimension by

$$\dim(0, 0, \ldots, x^i, \ldots, 0) = 2\dim x^i - (k-i) \quad \text{for } x^i \text{ in } L_{s-1}^i.$$

Then α in (5) is just the inclusion of $\Sigma_{i=0}^{k-2}(L_{s-1}^iM)$ into $\Sigma_{i=0}^{k-1}(L_{s-1}^iM)$, while β is just the projection of $\Sigma_{i=0}^{k-1}(L_{s-1}^iM)$ onto $L_{s-1}^{k-1}M$. If we ignore grading, L_s^kM is just the direct sum of $\binom{k}{s}$ copies of M. Our main result is

THEOREM 3. *For all $k \geqslant 0$, $s \geqslant 0$, $a \geqslant 0$ there are natural Z_2-*homomorphisms *d_s: $L_s^kM \longrightarrow L_{s-1}^kM$, $\lambda_s(a)$: $L_s^kM \longrightarrow L_s^kM$ with the following properties*:

(a) *$d_{s-1}d_s = 0$ so that L^kM is a chain complex. α, β in (5) are chain maps.*

(b) *The operators $\lambda_s(a)$ satisfy Adem relations "up to homotopy": if $b > a$ there are Z_2-linear maps $\beta_s(b, a)$: $L_s^kM \longrightarrow L_{s+1}^kM$ such that*

$$\lambda_s(b)\lambda_s(a) - \sum_{j\leqslant b/2} \binom{j-1}{a-b+2j} \lambda_s(b-2j)\lambda_s(a+j)$$
$$= d_{s+1}\beta_s(b, a) + \beta_{s-1}(b, a)d_s.$$

(c) *$d_s\lambda_s(a) = \lambda_{s-1}(a)d_s$, so that $\lambda_s(a)$ can be regarded as an operator on $H_s(L^kM)$.*

(d) *The operations $\lambda_s(a)$ vanish on $H_s(L^kM)$ if $a < k$, and $H_s(L^kM)$ becomes an unstable A-module if we put $\mathrm{Sq}_a = \lambda_s(a+k)$ for all $a \geqslant 0$.*

(e) *There is a natural isomorphism of unstable a-modules* $H_s(L^kM) = \Omega_s^kM$, *and the long exact sequence in homology associated with* (5) *is identical with* (3).

Details of this construction and applications to the computation of Ω_s^kM will appear elsewhere. We mention only that if S^n is the unique $\mathcal{M}_A$ object for which $(S^n)_n = Z_2$, $(S^n)_j = 0$ if $j \neq n$, then the differential d_s: $L_s^kS^n \longrightarrow L_{s-1}^kS^n$ vanishes if $k \leqslant n + s - 1$. This fact permits us to determine completely the unstable A-modules $\Omega_s^kS^n$ for those cases in which $k \leqslant n + s - 1$. For example, it turns out that $\Omega_s^{n+s-1}S^n$ is the suspension of a truncated polynomial algebra over A of a kind already classified by Sugawara and Toda in [4].

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