

of the coefficients of a cusp form and he applies it to show that the divisor function is a reasonably good approximation to the number of representations of n by the quadratic form. Because of the restriction to $N=1$ Schoeneberg does not discuss the classical theta function, $\vartheta(\tau) = \sum_{m=-\infty}^{\infty} \exp(\pi i m^2 \tau)$, which is of level $N=2$. $\vartheta^s(\tau)$ arises from the quadratic form $x_1^2 + \cdots + x_s^2$ and thus serves as a generating function for $r_s(n)$, the number of representations of n as a sum of s squares. The author's omission of $\vartheta(\tau)$ is unfortunate in a book of this size and scope, especially since he has developed all the machinery necessary to discuss $\vartheta^s(\tau)$, at least for $s \equiv 0 \pmod{8}$, in which case $\vartheta^s(\tau)$ is a modular form of level 2 and dimension $-s/2$, without multipliers.

Undoubtedly there will be some who view Schoeneberg's book as old-fashioned. Indeed, except for Chapter 8, the book could have been written in 1939, and even the 1967 article of the author, upon which Chapter 8 is based, conceivably could have been written in 1940. My own feeling is that we should be grateful for works of this quality whenever they appear. On the other hand, I regard as a flaw Schoeneberg's failure to introduce Hecke operators or the Petersson inner product. Were the book not otherwise excellent, these omissions, in themselves, would be no cause for concern. As things are, the first six chapters constitute a well-written, solid treatment of the classical theory of modular functions of a single variable, except for the omission of these two important topics. These chapters, together with appropriately chosen additional material, could serve as an excellent year-long introduction to the subject for graduate students with a reasonable background in analysis and algebra. It is a pity that additional material must be introduced for this purpose. The book is good enough that I cannot help feeling it could have been even better, and wishing it were.

The translation, by J. R. Smart and E. A. Schwandt, is generally smooth and free of awkward phrasings. Happily, it reads like English, with little, if any, trace of the original German detectable. I noticed several misprints, but a remarkably small number for a book of this length.

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The theory of stochastic processes. I, by I. I. Gihman and A. V. Skorohod, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 210, Springer-Verlag, New York, Heidelberg, Berlin, 1974, viii+570 pp., \$52.90.

Stochastic calculus and stochastic models, by E. J. McShane, Academic Press, New York, 1974, x+239 pp., \$19.50.

Mesures cylindriques, espaces de Wiener, et fonctions aléatoires Gaussiennes, by Albert Badrikian and Simone Chevet, Lecture Notes in Mathematics, No. 379, Springer-Verlag, Berlin, 1974, x+383 pp., \$13.20.

Almost sure convergence, by William F. Stout, Academic Press, New York, 1974, x+381 pp., \$24.50.

Integration in Hilbert space, by A. V. Skorohod, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 79, Springer-Verlag, New York, 1974, xii+177 pp., \$19.70.

It is appropriate that the five books under review, all published in the last couple of years and all on topics in probability theory, should reflect the diversity which exists today in the field. Of the books, *Almost sure convergence*, by William F. Stout, is by far the most classically oriented. The topic of "almost sure" phenomena goes back to the beginnings of the modern era in probability theory and remains one of the most intriguing. Starting with the well-known almost sure convergence results about martingales and partial sums of independent random variables, Stout's book covers, in great detail, those questions about almost sure convergence which have been handled by probabilistic, as opposed to other more purely analytic, techniques. In doing so, Stout has put into context a good deal of formerly unorganized material which has been accessible only to the *cognoscenti*. For instance, he devotes a good deal of space to martingale transforms and the ideas of Burkholder and Gundy on the theory of martingales. Also, he has included a complete rundown on the modern theory of the law of the iterated logarithm, including Strassen's version and Strassen's converse. Throughout, Stout's exposition is consistently clear. He has succeeded in presenting quite complicated and intricate proofs in a rigorous, yet uncluttered, way; and his commentary between theorems is both helpful and interesting.

Unfortunately, in spite of the quality of its presentation, it is somewhat difficult to guess to what audience Stout has addressed his book. The beginner in probability theory would be ill-advised to spend the time it requires to do justice to the three hundred and fifty odd pages which Stout has devoted to this rather esoteric subject. The expert, on the other hand, is probably aware of most of the material contained in this volume, although he may be grateful for the care with which Stout has organized it. The idea of building a book around "almost sure" phenomena is a novel one and deserved being tried; but the experiment might have been more useful had the author used his expertise in the subject to select a few pearls from the area and make them available to the general mathematical public rather than produce a treatise which, by its very weight, will appeal most to those who need it least.

The other four books deal with various aspects of the theory of stochastic processes. There are two main schools at present in this theory: the *path school* and the *measure school*. The book of E. J. McShane is firmly in the path school tradition; the one by I. I. Gihman and A. V. Skorohod straddles the two schools, although it definitely leans towards the measure school; and the book of A. Badrikian and S. Chevet and that by A. V. Skorohod (alone) are representative of a special branch of the measure school. The difference between the path and measure schools is, in large part, one of emphasis. For

a member of the path school, a stochastic process is determined by its paths. That is, to him the sample space on which the stochastic process is realized is nearly irrelevant, and the important properties of the process reside in some sort of equation satisfied by the paths. In contrast to this point of view, the measure school gives paramount importance to the measure and sees the paths as merely artifacts. A reasonable (but not infallible) litmus for testing an individual's affiliation is to remark whether he is more comfortable talking about Brownian motion or Wiener measure. (Of course, the happiest individuals are those who are comfortable talking about both.)

The major appeal of the path school, and the one emphasized by McShane in *Stochastic calculus and stochastic models*, is that it provides the most natural setting for certain problems in mathematical engineering. Many noise and filtering problems can be modeled mathematically by stochastic integral equations. For example, often what is being modeled is a situation in which a random signal, with known characteristics, is sent through a black box. Since, in practice, it is the signal which is observable, it is only the signal and the stochastic integral equation plays the role of the black box. Since, in practice, it is the signal which is the observable, it is only natural that the path should be the primary object of consideration in the mathematical model. Hence the interest in the path by path analysis of stochastic differential equations. There are three aspects of this topic to which McShane has addressed himself. First and foremost, there is a need for an integration theory which encompasses both Riemann theory and Itô theory at the same time. Secondly, from the practical point of view, it is important to know that the solution of a stochastic integral equation is pathwise stable under small changes in the driving process. Finally, one must understand the transformation properties of stochastic integral equations under changes of coordinates. In confronting the first of these problems, McShane has succeeded in getting away from an essentially martingale-based theory. He has done so at the expense of simplicity and, at times, elegance. On the other hand, his claim is that the martingale assumption is not very realistic in practice and therefore must be abandoned. Since there is no reason to doubt this claim, there is no way of denying the need for a theory with the generality that he enjoys. The only lingering doubt one might have is whether an equally satisfactory and more aesthetically pleasing theory might not have been developed using quasi-martingales, but this is only a quibble. Because he has freed himself from martingales, his stability theory is very strong, since his allowable perturbations are very general. This part of the book should be very appealing to the engineering community. Finally, the last chapter of the book explains the importance of what he has called equations in *canonical form*. What this turns out to be is the most desirable form in which to write a stochastic differential equation in order to maximize its invariance properties under changes of variables. For this reason, it is also the best way to write an equation if one wants to approximate its solutions by solutions of equations in which the driving signal has been replaced by a mollified version of itself. Because these are aspects of the theory which have been neglected by other Western authors

and because of the meticulous care with which McShane has treated them, his book is a valuable addition to the literature.

The theory of stochastic processes. I, by I. I. Gihman and A. V. Skorohod is the first in a three volume treatise on the subject. Not having seen what volumes II and III contain, the reviewer finds it difficult to say what these two prominent researchers in the field intended this work to be; but if they conceived of it as becoming the “Dunford and Schwartz” of stochastic processes, they have fallen short of their mark. Neither their organization nor their presentation is sufficiently well thought out to make the book a serious candidate for such a role. Indeed, since much of the material in the present book has already appeared in their *Introduction to the theory of random processes*, it is a little disappointing that they have not made a greater effort at polishing what they have done. Nevertheless, any book by these two is likely to contain much that is valuable, and this one is no exception.

It is unavoidable when discussing a general book on stochastic processes to see it in relation to the classic on this subject by J. L. Doob. In the present case, this is not hard to do. Doob's book was written at a period during which the path school was dominant, the book of Gihman and Skorohod is the product of men who have played an impressive part in the evolution of the measure school. Thus, although many of the topics are the same, the emphasis has changed in accordance with the predilections of the authors. As a consequence, much of Gihman and Skorohod's book is devoted to the theory and applications of weak convergence of measures on function spaces, a subject which had not even come into its own when Doob was writing. In particular, they have presented a modern proof of Donsker's invariance principle and related results. Presumably this groundwork will serve them well in the ensuing volumes. Another indication of the influence of the measure school on their tastes is the amount of space Gihman and Skorohod devote to the question of absolute continuity of stochastic processes. In fact, their discussion of this topic is as complete and up-to-date as there exists in any book. They have also given considerable room to the theory of measures on Hilbert space and some problems which naturally arise in this field. Much of the material included on this subject appears again in the book by Skorohod to be reviewed below.

Among the distinguishing characteristics of these two authors is their unflinching courage in the face of long and involved computations. The book under review is replete with detailed, and sometimes beautiful, calculations. Although the analysis used in these computations is classical and the tricks are well known to the founding fathers of modern probability, one cannot help but be impressed by Gihman and Skorohod's virtuosity. Also, for the descendants of the founding fathers, nourished as we have been on the formalism of the subject, these demonstrations of computational techniques may be the most valuable part of the book. It is sometimes refreshing to be reminded that our machinery sometimes leads to numbers.

In summary, it should be said that this book by Gihman and Skorohod

provides an interesting perspective on the theory of stochastic processes, and is not simply a rewriting of Doob's book. At the same time, it is, far too often, simply a rewriting of their own earlier book. Since the clarity of presentation, as distinguished from the price tag, has not been substantially increased in the rewriting process, it is doubtful if every serious probabilist will feel obliged to have this book on his shelf. Nonetheless, he will certainly be glad if his departmental library sees fit to have it on theirs.

It is now time to turn to *Integration on Hilbert space*, by A. V. Skorohod and *Mesures cylindriques, espaces de Wiener et fonctions aléatoires Gaussiennes*, by A. Badrikian and S. Chevet. (It is this reviewer's pleasure to thank G. Kallianpur for his permission to borrow his expert opinions in commenting on these books.) These books represent what might be called the "high church" of the measure school. Here, not only is the emphasis on the measure, but the paths have entirely disappeared. The idea is, in so far as possible, to get away from all finite dimensional considerations and describe, construct, and study the measure directly on a linear topological space (alias function space). Aside from the obvious Bourbakian appeal of this approach, the ideas evolved here have been motivated by and proved useful in the study of Euclidean quantum field theory, where there is no obviously distinguished parameter to use as the "time" variable.

The first problem that has to be faced in this theory is that of constructing a measure, with given properties, on the space. One of the best ways of prescribing the desired given properties is to specify the characteristic function of the measure that is eventually to be constructed. However, the situation here is no longer so simple as in finite dimensional spaces. Given a function on the dual space satisfying the naïve generalization of Bochner's criterion for characteristic functions, it is immediate that there exists a unique finitely additive measure on the space which is countably additive on finite dimensional subspaces and has the right characteristic function. The problem is that this measure need not be countably additive on the whole space. At this point, there are two avenues open to one. The first, which is due to Minlos and Sazonov, is to find the correct version of Bochner's theorem and see if it applies. This is the route chosen by Skorohod. If the first avenue leads to a dead end, then one must try the second, which is to abandon the original space and try to fit the measure on a larger one. The basic ideas behind this latter approach go back to the theory of Radonifying maps as well as the ideas of L. Gross. A major portion of Badrikian and Chevet's book is devoted to a detailed exposition of the form that Gross' ideas take after they have undergone thorough abstraction and generalization at the hands of the French school; in particular, an up-to-date account of cylinder measures is given along with applications to Gaussian measures and abstract Wiener spaces.

The difference in taste evidenced in their handling of the construction problem asserts itself at every point of comparison between these two books. Skorohod concentrates consistently on computationally tractable problems, while Badrikian and Chevet are more concerned with abstract

considerations. The topics covered by Skorohod are: measurable polynomials (this is an abstraction of the Itô-Wiener theory of homogeneous chaos in Wiener space); absolute continuity and quasi-invariance under shifts and nonlinear transformations; and surface integrals and Gauss' formula in Hilbert spaces. (The last of these topics appears here for the first time.) In contrast, the book of Badrikian and Chevet includes: GB and GC-sets, ϵ -entropy, a thorough discussion of the work of Sudakov with complete proofs (given for the first time) together with recent amplifications due to Chevet, and 0-1 phenomena and integrability properties of Gaussian measures. Of the two books, Badrikian and Chevet's is much more up-to-date and Skorohod's is much more accessible to the novice. Together they constitute a quite complete account of the state in which this art finds itself today; the one with its emphasis on computation, the other with its infatuation with generality and elegance. Unfortunately, neither one devotes any space to Feynman integration or the recent applications that this area has enjoyed in quantum field theory. On the other hand, if the success of these probabilistic techniques in physics continues, there will certainly be books forthcoming on that subject, and these books will then be appreciated for the groundwork which they have laid.

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Geometric theory of algebraic space curves, by S. S. Abhyankar and A. M. Sathaye, Lecture Notes in Mathematics, no. 423, Springer-Verlag, Berlin, Heidelberg, New York, 1974, xiv + 302 pp., \$11.50.

Let k be an algebraically closed field, for example $k = \mathbf{C}$ (the complex numbers) will do. An *affine algebraic variety* over k is the solution set of a family $\{f_\alpha(x_1, \dots, x_n)\}_\alpha$ of polynomials in n variables (for some n) with coefficients in k . Actually, we should be more precise about where our solutions are located. If A is a k -algebra (e.g., $A = k$ itself, or $A =$ some field extension of k) then we can evaluate the polynomials $f_\alpha(x_1, \dots, x_n)$ on n -tuples $\langle a_1, \dots, a_n \rangle$ from A . Hence, it makes sense to consider those n -tuples from A for which all the polynomials f_α vanish. These n -tuples are the points of our variety V with values in A (or rational over A). The whole variety, V , should be thought of as the collection of all the sets, $V(A)$, consisting of the points of V with values in A for all k -algebras A .

Classically, geometers considered only the case in which the k -algebra A was a field; since the book under review adopts a classical position, we shall also restrict attention to the case when A is a field. If Ω denotes an algebraically closed field of infinite transcendence degree over k , then it turns out that all phenomena of the classical variety V may be captured in the set $V(\Omega)$. We can therefore replace the somewhat nebulous idea of the collection $V(K)$ (where K is a field over k) by the one set $V(\Omega)$. Even more