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HIGHER WHITEHEAD GROUPS AND STABLE HOMOTOPY

BY JEAN-LOUIS LODAY

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ABSTRACT. Comparison of stable homotopy of B_G, algebraic K-theory of $\mathbf{Z}[G]$ and higher Whitehead groups of G. Computations of $\mathrm{Wh}_2(G)$.

In their work on pseudo-isotopy Hatcher and Wagoner [1] defined an obstruction group $\mathrm{Wh}_2(G)$ as follows. Let $\mathrm{GL}(\mathbf{Z}[G])$ (resp. $E(\mathbf{Z}[G])$, resp. $\mathrm{St}(\mathbf{Z}[G])$) be the general linear group (resp. its commutator subgroup, resp. the Steinberg group) of the group algebra $\mathbf{Z}[G]$. The kernel of the natural homomorphism $\mathrm{St}(\mathbf{Z}[G]) \longrightarrow E(\mathbf{Z}[G])$ is the group $K_2(\mathbf{Z}[G])$ defined by Milnor [4]. As usual $\widetilde{K}_2(\mathbf{Z}[G]) = \mathrm{Coker}(K_2(\mathbf{Z}) \longrightarrow K_2(\mathbf{Z}[G]))$.

If x_{ij}^a $(i \neq j,\ a \in \mathbf{Z}[G])$ are the classical generators of $\mathrm{St}(\mathbf{Z}[G])$ one denotes by $W(\pm G)$ the subgroup of $\mathrm{St}(\mathbf{Z}[G])$ generated by the elements $w_{ij}(\pm g) = x_{ij}^{\pm g} x_{ji}^{\mp g^{-1}} x_{ij}^{\pm g}$ $(g \in G)$. With these notations the second-order Whitehead group is

$$\mathrm{Wh}_2(G) = K_2(\mathbf{Z}[G])/K_2(\mathbf{Z}[G]) \cap W(\pm G).$$

Let B_G be the classifying space of the (discrete) group G. The nth stable homotopy group of B_G, $\pi_n^s(B_G) = \varinjlim \pi_{n+k}(S^k B_G)$ is also equal to the nth reduced homology group of B_G with coefficients in the sphere spectrum $\mathbf{S}$: $\widetilde{h}_n(B_G; \mathbf{S})$. Equivalently $\pi_n^s(B_G)$ is equal to $\pi_n(\Omega^\infty S^\infty B_G)$ where $\Omega^\infty S^\infty B_G = \varinjlim \Omega^k S^k B_G$.

THEOREM. *There exists a natural homomorphism* $\pi_2^s(B_G) \longrightarrow \widetilde{K}_2(\mathbf{Z}[G])$ *such that* $\mathrm{Wh}_2(G) = \mathrm{Coker}(\pi_2^s(B_G) \longrightarrow \widetilde{K}_2(\mathbf{Z}[G]))$.

Let $\mathbf{K}_A$ be the spectrum of algebraic K-theory associated to the (unitary) ring A (cf. [2], [6]). The homotopy groups $\pi_n(\mathbf{K}_A)$ are Quillen's K-groups denoted $K_n(A) = \pi_n(B^+_{\mathrm{GL}(A)})$, $n \geq 1$ (cf. [5]).

Let $h_*(-; \mathbf{K}_\mathbf{Z})$ be the generalised homology theory associated to $\mathbf{K}_\mathbf{Z}$. We construct natural maps of spectra

$$\mu\colon \mathbf{S} \longrightarrow \mathbf{K}_\mathbf{Z} \quad \text{and} \quad \lambda\colon B_G \wedge \mathbf{K}_\mathbf{Z} \longrightarrow \mathbf{K}_{\mathbf{Z}[G]}.$$

These maps give rise to the composed homomorphism

$$\pi_n^s(B_G \cup \mathrm{pt}) \xrightarrow{\mu_n} h_n(B_G; \mathbf{K}_\mathbf{Z}) \xrightarrow{\lambda_n} K_n(\mathbf{Z}[G]).$$

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Here $B_G \cup \mathrm{pt}$ denotes the disjoint union of B_G and a point. Recall that $\pi_n^s(B_G \cup \mathrm{pt}) = h_n(B_G; \mathbf{S})$. These constructions can be done either by means of the categorical approach of Anderson and Segal (cf. [6]) or in the framework of Quillen's "+" construction (cf. [3]). The proof of the theorem splits into two parts. First we prove that μ_2 is an isomorphism and then we compute the image of λ_2.

REMARK. We can show that $\lambda_n \circ \mu_n$ can be obtained, for $n \geqslant 1$, from a map $(\Omega^\infty S^\infty(B_G \cup \mathrm{pt}))_0 \longrightarrow B^+_{\mathrm{GL}(\mathbf{Z}[G])}$ by taking homotopy groups ($(X)_0$ is the connected component of the base-point of the space X).

Waldhausen [6] has proved that for a large class Cl of groups the homomorphism μ_n is an isomorphism. This result for $n = 2$ together with the theorem gives the following.

COROLLARY. *If G is in Cl the higher Whitehead group* $\mathrm{Wh}_2(G)$ *is trivial.*

Examples of groups in Cl:

$-G$ is the fundamental group of some submanifold of the 3-dimensional sphere,

$-G$ is a torsion-free one-relation group,

$-G$ is an iterated amalgamated sum (or HNN-extension) of free groups.

For the precise (and technical) definition of Cl we refer to [6].

The classical Whitehead group $\mathrm{Wh}_1(G) = K_1(\mathbf{Z}[G])/\pm G$ fits into the exact sequence

$$0 \longrightarrow \pi_1^s(B_G \cup \mathrm{pt}) \longrightarrow K_1(\mathbf{Z}[G]) \longrightarrow \mathrm{Wh}_1(G) \longrightarrow 0.$$

This sequence together with the theorem gives the exact sequence

$$\pi_2^s(B_G \cup \mathrm{pt}) \longrightarrow K_2(\mathbf{Z}[G]) \longrightarrow \mathrm{Wh}_2(G)$$
$$\longrightarrow \pi_1^s(B_G \cup \mathrm{pt}) \longrightarrow K_1(\mathbf{Z}[G]) \longrightarrow \mathrm{Wh}_1(G) \longrightarrow 0.$$

As $\pi_n^s(B_G \cup \mathrm{pt}) = \pi_n((\Omega^\infty S^\infty(B_G \cup \mathrm{pt}))_0)$ and $K_n(\mathbf{Z}[G]) = \pi_n(B^+_{\mathrm{GL}(\mathbf{Z}[G])})$ when $n \geqslant 1$, that last sequence looks like the lower part of the homotopy exact sequence of a fibration. Let F_G be the homotopy-theoric fiber of $(\Omega^\infty S^\infty(B_G \cup \mathrm{pt}))_0 \longrightarrow B^+_{\mathrm{GL}(\mathbf{Z}[G])}$ (see remark above).

PROPOSITION 1. *For every group G the equalities* $\mathrm{Wh}_1(G) = \pi_0(F_G)$ *and* $\mathrm{Wh}_2(G) = \pi_1(F_G)$ *hold.*

This suggests the following definition.

DEFINITION. *For every $n \geqslant 1$ and every group G the group $\pi_{n-1}(F_G)$ is called the higher Whitehead group (of order n) of G and is denoted by* $\mathrm{Wh}_n(G)$.

PROPOSITION 2. *For every group G in Cl,* $\mathrm{Wh}_3(G) = \mathrm{Wh}_3(0)$. *Moreover* $\mathrm{Wh}_3(0) = \mathrm{Coker}(\pi_3^s(S^0) \longrightarrow K_3(\mathbf{Z}))$.

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INSTITUT DE RECHERCHE MATHEMATIQUE AVANCÉE, UNIVERSITÉ LOUIS PASTEUR, STRASBOURG, FRANCE