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CONJUGATE SYSTEM CHARACTERIZATIONS OF H^1: COUNTER EXAMPLES FOR THE EUCLIDEAN PLANE AND LOCAL FIELDS

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ABSTRACT. The characterization of the Hardy space, H^1 of the plane, as those integrable functions whose first order Riesz transforms are (or whose maximal function is) integrable is well known. J.-A. Chao and M. Taibleson have shown that there is a conjugate system characterization of H^1 of a local field that parallels the Riesz system characterization of $H^1(R^2)$. C. Fefferman has conjectured that "nice" conjugate systems, such as the second order Riesz transforms would also give a characterization of $H^1(R^2)$. In the present paper a counter example of A. Gandulfo and M. Taibleson is described that shows that any conjugate system generated by an even kernel will fail to characterize H^1 of a local field. A counter example of J. Garcia-Cuerva is described that shows that the second order Riesz system for the Euclidean plane (which is generated by an even kernel) will fail to characterize $H^1(R^2)$ in the above sense.

Let $f \in L^1(\mathbf{R}^n)$ and let $f^*(x) = \sup_{y>0}|f(x, y)|$, where $f(x, y)$ is the Poisson integral of f. We say that $f \in H^1(\mathbf{R}^n)$ iff $f^* \in L^1(\mathbf{R}^n)$. Let (r, θ) be the polar representation of $(x_1, x_2) \in \mathbf{R}^2$, and let $(\cdot)^\wedge$ and $(\cdot)^\vee$ represents the Fourier transform and its inverse. The following characterization of $H^1(\mathbf{R}^2)$ is in [5, §8]:

THEOREM A. *If f is real-valued and $f \in L^1(\mathbf{R}^2)$, then $f \in H^1(\mathbf{R}^2)$ iff $(e^{i\theta}\hat{f})^\vee \in L^1(\mathbf{R}^2)$.*

Similarly, if K is a local field, e.g., a p-adic field, we may define $f^*(x) = \sup_{k\in \mathbf{Z}}|f(x, k)|$, where $f(x, k)$ is the regularization of f. (See [6, Chapter IV].) We say that $f \in H^1(K)$ iff $f^* \in L^1(K)$. The following characterization of $H^1(K)$ follows from results of Chao and Taibleson [3] and Chao [1], [2].

THEOREM B. *Suppose π is a multiplicative character on K that is unitary, ramified of degree 1, homogeneous of degree 0 and odd. If $f \in L^1(K)$ then $f \in H^1(K)$ iff $(\pi\hat{f})^\vee \in L^1(K)$.*

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The "only if" part of the proof is in Chao [2]. The "if" part follows from [3, Theorem 2] and [2, Theorem 3.1 and example (i), p. 282].

The "if" part of the proof of Theorem B depends on the fact that π is an odd function. Taibleson and Gandulfo investigated this point and have shown Theorem B fails if π is even.

THEOREM 1. *Suppose λ is a multiplicative character on K that is unitary, ramified of degree* 1, *homogeneous of degree* 0 *and even. Then, there is a real-valued function g, $g \in L^1(K)$ such that $\lambda\hat{g} = \hat{g}$ and $g^* \notin L^1(K)$.*

Thus, g and $(\lambda\hat{g})^\vee \in L^1(K)$ but $g \notin H^1(K)$. If the local class field of K is odd and of order not equal to 3 (e.g., a p-adic field with $p \neq 2$ or 3) then there is a character π on K that satisfies the conditions of Theorem B while π^2 satisfies the conditions of Theorem 1. Note that $f \longrightarrow (\pi^2\hat{f})^\vee$ is bounded from H^1 into itself (Chao [2]).

This result suggested that a similar investigation be made of the multiplier $e^{2i\theta}$ on $\mathbf{R}^2$. Note that $f \longrightarrow (e^{2i\theta}\hat{f})^\vee$ is bounded from H^1 into itself (Fefferman and Stein [5, p. 190]). Recently it has been conjectured by Fefferman [4] that any "nice" multiplier should characterize H^1 in the sense of Theorem A. In particular, $e^{2i\theta}$ is a usual example of such a "nice" multiplier. Garcia-Cuerva has investigated this problem and obtained the following result:

THEOREM 2. *There is a real-valued, radial function g, $g \in L^1(\mathbf{R}^2)$ such that $(e^{2i\theta}g)^\vee \in L^1(\mathbf{R}^2)$ but $g \notin H^1(\mathbf{R}^2)$.*

We now briefly sketch proofs of Theorems 1 and 2.

LEMMA 1. *Let λ be as in Theorem* 1. *Then there exists a finite Borel measure μ, supported on $\mathfrak{O}$ (the ring of integers in K) such that μ is singular, $\mu(\mathfrak{O}) = 0$ and $\lambda\hat{\mu} = \hat{\mu}$.*

Theorem 1 follows from Lemma 1. We note that $\mu^* \notin L^1$, where $\mu^*(x) = \sup_k |\mu(x, k)|$. Also $\sup_k \|\mu(\cdot, k)\|_1 < \infty$. Using the fact that $\mu(x, k)$ is supported on $\mathfrak{O} \times \mathbf{Z}$ we define $f(x) = \Sigma_{k=-\infty}^{1} a_k \mu(x + c_k, k)$ where $\{c_k\}$ are coset representatives of $\mathfrak{O}$ in K. If $\Sigma |a_k| < \infty$ we see that $f \in L^1(K)$ and $\lambda\hat{f} = \hat{f}$. Note that $(\mu(\cdot + c_k, k))^*(x) = \sup_{l \geq k} |\mu(x, l)|$. Thus, $\sup_k \|\mu(\cdot + c_k, k)^*\|_1 = \infty$, and we may choose the $\{a_k\}$ so that $f^* \notin L^1(K)$.

To construct the measure μ we need to construct a regular function $\mu(x, k)$ on $K \times \mathbf{Z}$ such that $\mu(x, k)$ is supported on $\mathfrak{O} \times \mathbf{Z}$, $\int_{\mathfrak{O}} \mu(x, k)dx = 0$ for all k, $\|\mu(\cdot, k)\|_1 \leq A$ and $\|\mu(\cdot, k) - \mu(\cdot, k-1)\|_1 = B$, $k = -1, -2, \ldots$, for positive constants A and B. (See [6, IV(1.8d) and (1.9b)].)

One now observes that if χ is an additive character on K that is nontrivial on $\mathfrak{O}$, but is trivial on $\mathfrak{P}$ (the maximal ideal in $\mathfrak{O}$) then

$$g(x) = \begin{cases} \operatorname{Re} \chi(x), & x \in \mathfrak{O}, \\ 0, & x \notin \mathfrak{O}, \end{cases}$$

has the property that $\lambda\hat{g} = \hat{g}$ whenever λ is as in Theorem 1, $\mu(x, k)$ is constructed by "patching together" various translations and dilations of g.

For a sketch of the proof of Theorem 2 we will identify $\mathbf{R}^2$ with $\mathbf{C}$ in the usual way: $(x_1, x_2) \longleftrightarrow re^{i\theta} = z$.

For $f \in L^1(\mathbf{C})$ let $\widetilde{f}(w) = \text{P.V.}\int_{\mathbf{C}} f(w - z)\,dz/\bar{z}^2$. Then, $(\widetilde{f})\hat{} = e^{2i\theta}\hat{f}$. We now assume that f is radial; i.e., $f(z) = g(|z|)$ for some g. We then show that if f is radial on $\mathbf{C}$ then $f \in H^1(\mathbf{C})$ iff $rg(r) \in H^1$ where $rg(r)$ can be viewed as either a function defined on $[0, \infty)$ or as an even function on $\mathbf{R}$. Finally we show that

$$\widetilde{f}(re^{i\theta}) = \pi e^{2i\theta}\left\{\frac{2}{r^2}\int_0^r g(s)\,s\,ds - g(r)\right\}.$$

Thus, we see that we need to find a function $\varphi \in L^1(0, \infty)$ such that $(1/r)\int_0^r \varphi(s)\,ds \in L^1(0, \infty)$ but $\varphi \in H^1(0, \infty)$.

Let $I_{[a,b]}$ be the characteristic function of the interval $[a, b]$, and let $l_k = k\,I_{[k,k+1/k]} - (1/k)I_{[k+1/k,2k+1/k]}$. We see that $\int_0^\infty |l_k| = 2$, $\int_0^\infty l_k = 0$, $\int_0^\infty |(1/r)\int_0^r l_k|\,dr \leqslant 1$. We see that there is a $C > 0$ such that if k is large enough $\int_{k/2}^k |\widetilde{l}_k|\,dr \geqslant C \ln k$. A little calculation shows that if n_0 is large enough, then $\varphi = \sum_1^\infty (1/n^2)\, l_{n_0 6^n}$ has the required properties.

As a final comment, we observe that the formula for $\widetilde{f}$, f integrable and radial extends easily to finite Borel measures that are radial. Apply that result to the singular measure μ that has measure 1 uniformly distributed on the unit circle in $\mathbf{C}$ and measure -1 uniformly distributed on the circle of radius two. It is easy to check that $\widetilde{\mu}$ is a singular measure. Together with the result of Lemma 1 we see that the conjugate systems induced by the multipliers $e^{2i\theta}$ and π^2 (on the Euclidean plane or local fields respectively) fail to produce an F. and M. Riesz theorem in the sense: There is a finite Borel measure μ, such that the conjugate of μ is also a finite measure, but μ is not absolutely continuous.

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