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ON PRIMARY BANACH SPACES

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A Banach space X is called primary (resp. prime) if for every projection P on X, PX or $(I - P)X$ (resp. PX with $\dim PX = \infty$) is isomorphic to X. It is well known that c_0 and l_p, $1 \leqslant p \leqslant \infty$, are prime spaces [5], [8], but it is an open question whether there are other prime Banach spaces. However, it is known that $C[0, 1]$ [7] and $L_p[0, 1]$, $1 < p < \infty$ [1], are primary, and in the recent Special Seminar on Functional Analysis at Urbana, Illinois, August, 1975, it is announced [2] that $C(K)$ is primary for any countable compact metric space K. For a discussion on prime and primary Banach spaces, we refer to [6].

For a Banach sequence space $(E, \|\cdot\|_E)$ and a sequence of Banach spaces $\{X_n\}$, we shall let $(X_1 \oplus X_2 \oplus \cdots)_E$ be the Banach space of all sequences $\{x_n\}$ such that $x_n \in X_n$, $n = 1, 2, \ldots$ and $(\|x_1\|, \|x_2\|, \ldots) \in E$ with the norm $\|\{x_n\}\| = \|(\|x_1\|, \|x_2\|, \ldots)\|_E$.

A basis $\{e_n\}$ in a Banach space X is called symmetric (cf. [10]) if every permutation $\{e_{\pi(n)}\}$ of $\{e_n\}$ is a basis of X, equivalent to $\{e_n\}$. For a basis $\{e_n\}$ of a Banach space X, we shall let X_n be the linear span of $e_1, e_2, \ldots, e_n$ in X.

MAIN THEOREM. *Let X be a Banach space with symmetric basis $\{e_n\}$. Then the following spaces are primary.*

(i) $(X \oplus X \oplus \cdots)_{l_p}$, $1 < p < \infty$, *where X is not isomorphic to* l_1.

(ii) $(X_1 \oplus X_2 \oplus \cdots)_{l_p}$, $1 < p < \infty$, *and* $(X_1 \oplus X_2 \oplus \cdots)_{c_0}$.

(iii) $(l_\infty \oplus l_\infty \oplus \cdots)_{l_p}$, $1 \leqslant p \leqslant \infty$, *and* $(l_\infty \oplus l_\infty \oplus \cdots)_{c_0}$.

Different techniques are needed in each of the three cases, and the cases $p = 1$ or when X is isomorphic to l_1 have to be treated separately. The proof for (i) is similar to the technique developed in [3]. To prove (ii), we use Ramsey's combinatorial lemma [9] and the following

LEMMA. *Let $M = \{m_i\}$ be a sequence of positive integers such that $\limsup m_i = \infty$. Then there exist rearrangements of M and the set of positive integers N into double sequences $\{m'_1, m'_2, \ldots; m''_1, m''_2, \ldots\}$ and $\{n'_1, n'_2, \ldots; n''_1, n''_2, \ldots\}$ such that $m'_i = n'_{2i-1} + n'_{2i}$ and $m''_{2i-1} + m''_{2i} = n''_i$, $i = 1, 2, \ldots$.*

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COROLLARY. *Let X be a Banach space with symmetric basis and let $\{B_n\}$ be a sequence of Banach spaces with* $\dim B_n = n$, $n = 1, 2, \ldots$. *If there exists a constant K such that the Banach-Mazur distance*

$$d(B_n \oplus B_m, B_{n+m}) = \inf\{\|T\|\|T^{-1}\|: T: B_n \oplus B_m \to B_{n+m} \text{ linear isomorphism}\} \leqslant K$$

for all $n, m = 1, 2, \ldots$, then $(B_1 \oplus B_2 \oplus \cdots)_X$ is isomorphic to $(B_{m_1} \oplus B_{m_2} \oplus \cdots)_X$ for all $\{m_i\}$ with $\limsup m_i = \infty$.

REMARK. When $X = l_p$, $1 < p < \infty$, a similar result was stated in [4, Lemma 5].

The proof of (iii) consists of generalizing the technique used by Lindenstrauss [5] in proving that l_∞ is prime and the following fact which is interesting in itself.

THEOREM 2. *Let X be a Banach space with symmetric basis. If E is a Banach space which has a complemented subspace isomorphic to X, then for any bounded linear operator T: $E \to E$, either TE or $(I - T)E$ has a complemented subspace isomorphic to X.*

By combining the techniques used to prove the Main Theorem and Theorem 2, we could obtain, for example,

THEOREM 3. *Let X be a Banach space with symmetric basis. If E is a Banach space which has a complemented subspace isomorphic to $(X \oplus X \oplus \cdots)_{l_p}$, $1 < p < \infty$ (resp. $(X \oplus X \oplus \cdots)_{c_0}$), then for any bounded linear operator $T: E \to E$, either TE or $(I - T)E$ contains a complemented subspace isomorphic to $(X \oplus X \oplus \cdots)_{l_p}$, $1 < p < \infty$ (resp. $(X \oplus X \oplus \cdots)_{c_0}$).*

Details of proofs will appear elsewhere.

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