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HYPERBOLIC EQUATIONS AND GROUP REPRESENTATIONS

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I. **Introduction.** In the study of eigenfunction expansions for a differential operator D one usually introduces the resolvent of D or, what is essentially the same thing, the heat operator related to D. This means that we study an operator involving more variables than D and then "descend" to D itself. An analogous idea is employed in case D is the Laplacian on the sphere. The eigenfunction theory for D is derived from the study of the Laplacian in the whole euclidean space; "descent" is separation of variables.

Our ideas can be thought of as an extension of separation of variables. Suppose we are given a homogeneous space $V = G/H$ of G. We want to decompose the representation of G on $L_2(V)$ (if V has an invariant measure) or on other function spaces on V. In rough terms, this is the problem of simultaneous eigenfunction expansion for the operators in the enveloping algebra of G which commute with H. The introduction of more variables is accomplished by finding a finite dimensional representation ρ of G which has an orbit which is G/H. ρ must be "suitable" in order that we can find a system of differential equations in the whole representation space which descends properly to this orbit. In what follows we illustrate the theory.

II. **Hyperbolicity and symmetric spaces.** Let G be a real semisimple Lie group in Chevalley (normal) form and let $\rho_1, \ldots, \rho_r$ be its fundamental representations. We set $\rho = \rho_1^2 \oplus \ldots \oplus \rho_r^2$. Now for each i there is a point u_i in the representation space of ρ_i^2 which is fixed exactly by K. All other points which are fixed under ρ_i^2 by K are of the form $t_i u_i$ where t_i is a scalar. We call $\{t_1 u_1 + \ldots + t_r u_r\} = T$ the *time axis* in analogy to the case $G = \mathrm{SL}(2, R)$. T is the set of K fixed vectors.

We call v_i the highest weight vector for each ρ_i^2 and we set $v = \Sigma v_i$. Then the isotropy group of v is MN. We call $\rho(G)v = \Gamma^+$ the *positive light cone.* The real algebraic closure of Γ^+ is denoted by Γ and is called the *light cone.* Note that A normalizes MN so A acts on Γ^+. This action of A coincides with scalar multiplication, a fact which is crucial in what follows.

Another important property of ρ is that both $\rho(G)u \approx G/K$ and $\Gamma^+ \approx G/MN$ appear in the same representation space. Thus we can study relations be-

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tween the representation theory on G/K and the representation theory on G/MN (which is simple). G/MN and G/K can be related even more closely by the following geometric property of ρ. Denote by RGu the region swept out by applying scalar multiplication to $\rho(G)u$. Then as in the case of SL(2, R),

PROPOSITION 1. *Γ^+ lies in the closure of RGu.*

We can construct a G invariant nondegenerate bilinear form on the space of ρ which we denote by $x \cdot \hat{x}$. In terms of this we have a notion of Fourier transform. We shall denote by $\hat{\Gamma}$, $\hat{T}$, etc. the analogs of Γ, T, etc. in $\hat{x}$ space. Now, $\hat{\Gamma}$ is an algebraic variety so is defined by polynomial equations $P_i(\hat{x}) = 0$ which we can construct explicitly. Under Fourier transform these lead to differential operators $P_j(i\partial/\partial x)$. We denote by $\partial(\Gamma)$ this set of operators. It is $\partial(\Gamma)$ which is the system in many variables described in I above which has the proper descent to $\rho(G)u$.

THEOREM 2. *$\partial(\Gamma)$ is hyperbolic with space-like surface $\rho(G)u$. More precisely, we can set up a well-posed Cauchy Problem for $\rho(\Gamma)$ where we give $w =$* order *W data on $\rho(G)u$.*

REMARK. In case G is not in Chevalley form we can make an analogous construction. However $\partial(\Gamma)$ is hyperbolic only for Chevalley forms. In fact if G is compact the analog of $\partial(\Gamma)$ is elliptic.

III. **Energy theory**. The ideas that go into Theorem 2 are as follows:

A. Using the theory of harmonic functions for the Weyl group W, which works because W is generated by reflections, we can find an *energy* $\epsilon(f)$ for solutions f of $\partial(\Gamma)f = 0$ near RGu. The energy is an integral along $r\rho(G)u$ for any scalar r, of a quadratic form in G derivatives of the Cauchy data of f on $r\rho(G)u$. The energy is independent of r. However, we cannot show directly that it is positive definite.

B. *Cauchy-Koweleski theory.* It is not difficult to show that the complex-analytic Cauchy Problem for $\partial(\Gamma)$ is well posed on $\rho(G)u$.

C. *Limit on* Γ. Using Proposition 1 and A above we show that, for a dense set of Cauchy data, the solution f has a limit on Γ. The limit of $\epsilon(f)$ can be expressed in terms of scalar derivatives of f on Γ. Using Fourier analysis it follows that this limit is $\geqslant 0$. Thus $\epsilon(f) \geqslant 0$.

In order to know that $\epsilon(f) \gg 0$ we need

D. *Uniqueness of Dirichlet Problem.* Every suitable solution f of $\partial(f) = 0$ is determined by its restriction to Γ. This is proven by writing an explicit formula for f in terms of its Dirichlet data. This generalizes d'Adhemard's formula for the wave equation.

Further consequences of the above are

E. *Plancherel formula for $\rho(G)u$.* On studying the limit in C and using the positive definiteness of $\epsilon(f)$ we can compute the Plancherel measure for $\rho(G)u$.

F. *Uniqueness and Paley-Wiener theory.* Combining energy theory with B shows that there is a domain of dependence for the Cauchy Problem. By general principles this uniqueness property implies the Paley-Wiener theorem for $\rho(G)u$, hence for G/K.

G. *Orbital integrals.* From $\epsilon(f)$ we derive a bilinear form $\epsilon(f, g)$ on solutions of $\partial(\Gamma)$ which is constant on each of the orbits $r\rho(G)u$. Choosing g suitably we find that the integral of f is constant on all $r\rho(G)u$. Using C it has the same value on Γ^+. This generalized the known mean-value theory for compact groups.

IV. **Other representations of** G.

H. *Parabolic subgroups.* Instead of using ρ we could use other combinations of the fundamental representations of G. We can do this in such a way that any parabolic subgroup P of G can (in terms of its Levi splitting) take the place of the minimal parabolic used in II, III.

I. *Discrete series.* Instead of studying the Cauchy Problem we can study the Watergate Problem which means that we parametrize solutions f of $\partial(\Gamma)$ by data on the time axis T. (We must give infinitely many data.) We show that this leads to a well-posed problem. If all the Watergate data are concentrated at $t = 0$, then f vanishes on RGu. Moreover f is small on other orbits of $\rho(G)$. f is actually small in $r_0 = \text{rank } K$ "directions". If $r_0 = r$ then the restriction of f to these orbits belongs to the discrete series. In fact, all "generic" discrete series can be constructed in this manner.

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