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ON A MEAN VALUE INEQUALITY

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In this note we discuss a mean value inequality satisfied by functions $u(x, t)$ defined in the half space R^{n+1}_+ which are solutions of a partial differential equation of semielliptic type. We then apply this result to the study of spaces of non-isotropic Riesz potentials and to the determination of the classes which arise as traces of the functions $u(x, t)$. The justification for considering these functions lies in the fact that they are a natural substitute for harmonic functions when Laplace's equation is not satisfied and they are related to the study of singular integrals with mixed homogeneity. It is a pleasure to acknowledge the conversations we had with Dr. A. P. Calderón concerning these topics.

The mean value inequality. We let $\{A_t\}_{t>0}$, $A_{ts} = A_tA_s$ be a continuous group of affine transformations of R^n leaving the origin fixed and denote its infinitesimal generator by P so that $t(d/dt)A_t = PA_t$. We further assume that $(Px, x) \geqslant (x, x)$ for $x \in R^n$ and associate to each group A_t a translation invariant distance function $\rho(x)$ defined to be the unique value of t such that $|A_t^{-1}x| = 1$, $\rho(0) = 0$. To the transpose A_t^* of A_t we associate $\rho^*(x)$ in a similar fashion. As is well known $\det A_t = \det A_t^* = t^\gamma$, $\gamma = \operatorname{trace} P$ (see [5, §1.1]). For $\alpha = (\alpha_1, \ldots, \alpha_k)$, $1 \leqslant \alpha_i \leqslant n$, and $x^1, \ldots, x^k$ in R^n we let $\zeta = x^1 \otimes \cdots \otimes x^k$ to be the element with components $\zeta_\alpha = \prod_{i=1}^k x^i_{\alpha_i}$. For $n \times n$ matrices $A_1, \ldots, A_k$, we put $(A_1 \otimes \cdots \otimes A_k)(x^1 \otimes \cdots \otimes x^k) = A_1x^1 \otimes \cdots \otimes A_kx^k$ and abbreviate this by $\bigotimes^k Ax$ when $A_i = A$, $x^i = x$ for $1 \leqslant i \leqslant k$.

$\partial = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, $\partial/\partial t$ and $\bigotimes^k A\partial$ acting on functions $u(x, t)$ have the obvious meaning. We set $p_k(t, \partial) = \bigotimes^k LA_t^*\partial$, where $L^2 = (P + P^*)/4\pi$. Given a function $\psi(x)$ we define the dilations $\psi_t(x) = t^{-\gamma}\psi(A_t^{-1}x)$. A special role is played by $\varphi_t(x)$ with $\varphi(x) = e^{-\pi|x|^2}$. This particular function $\varphi_t(x)$ satisfies a differential equation, as is readily seen by taking Fourier transforms, namely $A\varphi_t(x) = 0$ where

$$A = \frac{\partial}{\partial t} - \frac{1}{2\pi t}(P^*A_t^*\partial, A_t^*\partial) = \frac{\partial}{\partial t} - \frac{1}{t}(LA_t^*\partial, LA_t^*\partial).$$

We also have $Au(x, t) = 0$, whenever $u(x, t) = f_*\varphi_t(x)$, $f \in S'(R^n)$.

We now state the mean value inequality and give some applications in the following sections.

MEAN VALUE INEQUALITY. Let $Au(x, t) = 0$ and $0 \leqslant r \leqslant k$, then

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$$|p_k(t, \partial)u(x, t)|^q \leqslant ct^{-\gamma}\int_{t/2}^{t}\int_{\rho(x-y)\leqslant t}|p_r(s, \partial)u(y, s)|^q \, dy \, \frac{ds}{s},$$

for $1 \leqslant q < \infty$.

Nonisotropic Riesz potentials. (See [1], [3], [7], [12], [18] and [20].) For a positive real number α we define the Riesz potential I_α of order α of f by means of

$$(I_\alpha f)\hat{}\,(x) = \rho^*(x)^{-\alpha}\hat{f}(x), \qquad 0 < \alpha < \gamma,$$

and for $1 < p < \infty$, the classes $L^p_\alpha(R^n) = \{f \in L^p(R^n): f = I_\alpha \eta, \eta \in L^p(R^n)\}$ and we set $\|f\|_{p,\alpha} = \|f\|_p + \|\eta\|_p$.

We now consider the following variants of the Littlewood-Paley function to express the norm in L^p_α by an equivalent quantity (see [4], [10], [14], [16], [17]). Let

$$G_q(k, \alpha, \lambda, x) = \left[\int_0^\infty \int_{R^n} \frac{|p_k(s, \partial)u(y, s)|^q}{(1 + \rho(x - y)/s)^{\gamma\lambda}} s^{-\alpha q - \gamma} \, dy \, \frac{ds}{s}\right]^{1/q}$$

for $k \geqslant 1$, $0 < \alpha < k$ and $\lambda > 1$.

THEOREM. *Let $u(x, t) = f_* \varphi_t(x)$; then f is in $L^p_\alpha(R^n)$ if and only if f is in $L^p(R^n)$ and $G_2(k, \alpha, \lambda, x)$ is in $L^p(R^n)$, provided $\lambda > 2/p$, and $\|f\|_{p,\alpha} \approx \|f\|_p + \|G_2(k, \alpha, \lambda)\|_p$. Moreover, if $q \geqslant 2$ and $\lambda > q/p$, then $\|G_q(k, \alpha, \lambda)\|_p \leqslant c\|f\|_{p,\alpha}$, and if $\lambda = q/p$ and $p < 2$ we have the weak-type inequality*

$$|\{x \in R^n: G_q(k, \alpha, q/p, x) > \mu\}| \leqslant c\|f\|^p_{p,\alpha}/\mu^p.$$

That such weak-type inequalities follow from results in [5, §3.3] was indicated to us by N. Aguilera.

Closely related to these questions are the functions $\mathcal{D}^\alpha_q$ and $\mathcal{D}^\alpha_{p,q}$ (see [2], [14], [15], [20]) defined as follows:

$$\mathcal{D}^\alpha_q(x) = \left[\int \frac{|f(x - y) - f(x)|^q}{\rho(y)^{\gamma + \alpha q}} \, dy\right]^{1/q},$$

$$\mathcal{D}^\alpha_{p,q}(x) = \left[\int_0^\infty \frac{1}{t^{\alpha q}} \left\{\int_{\rho(y)\leqslant 1} |f(x + A_t y) - f(x)|^p \, dy\right\}^{q/p} \frac{dt}{t}\right]^{1/q}$$

where $0 < \alpha < 1$ and $1 \leqslant p, q < \infty$.

Indeed we have the following result.

THEOREM. *Let $u(x, t) = f_* \varphi_t(x)$; then for $p > 2\gamma/(\gamma + 2\alpha)$,*

$$\|f\|_{p,\alpha} \approx \|f\|_p + \|\mathcal{D}^\alpha_2\|_p \quad \text{and} \quad |\{x \in R^n: \mathcal{D}^\alpha_2(x) > \mu\}| \leqslant c\|f\|^p_{p,\alpha}/\mu^p$$

for $p = 2\gamma/(\gamma + 2\alpha)$.

Also if $1 \leqslant r \leqslant q < \infty$, $q \geqslant 2$ and $p > r\gamma/(\gamma + \alpha r)$, then

$$\|\mathcal{D}^{\alpha}_{rq}\|_p \leq c\|f\|_{p,\alpha}, \quad \text{and} \quad |\{x \in R^n: \mathcal{D}^{\alpha}_{rq}(x) > \mu\}| \leq c\|f\|^p_{p,\alpha}/\mu^p,$$

for $1 < p = r\gamma/(\gamma + \alpha r) < 2$.

Traces of the spaces $H^{\alpha,p}$. These results were obtained jointly with A. Ortiz and extend the interesting results of [6]. Let $0 \leq \alpha < 1$, $1 \leq p < \infty$. We say that $u(x, t) \in H^{\alpha,p}$ if $Au(x, t) = 0$ and

$$|u|_{p,\alpha} = \sup_{x,t}\left[\frac{1}{t^{\gamma+\alpha p}}\int_{\rho(x-y)\leq t}\left[\int_0^t |p_1(s, \partial)u(y, s)|^2 \frac{ds}{s}\right]^{p/2} dy\right]^{1/p} < \infty.$$

Then the following holds.

THEOREM. *$u(x, t) \in H^{\alpha,p}$ if and only if $u(x, t) = f_*\varphi_t(x)$, where $f \in L^p_{\text{loc}}(R^n)$ and*

$$\sup_{x,t}\left[\frac{1}{t^{\gamma+\alpha p}}\int_{\rho(x-y)\leq t} |f(y) - av_{x,t}f|^p\, dy\right]^{1/p} < \infty,$$

where

$$av_{x,t}f = \frac{1}{|\{z: \rho(z) \leq t\}|}\int_{\rho(x-y)\leq t} f(y)\, dy.$$

Therefore the spaces of functions $f(x)$ which arise as traces of functions $u(x, t)$ in $H^{\alpha,p}$ are global Lipschitz classes for $0 < \alpha < 1$ and BMO for $\alpha = 0$.

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