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THE EULER CHARACTERISTIC OF AN AFFINE SPACE FORM IS ZERO

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An affine space form is a compact manifold obtained by forming the quotient of ordinary space R^n by a discrete group of affine transformations. An affine manifold is one which admits a covering by coordinate systems where the overlap transformations are restrictions of affine transformations on R^n. Affine space forms are characterized among affine manifolds by a completeness property of straight lines in the universal cover.

If $n = 2m$ is even, it is unknown whether or not the Euler characteristic of a compact affine manifold can be nonzero. The curvature definition of characteristic classes shows that all classes, except possibly the Euler class, vanish. This argument breaks down for the Euler class because the pfaffian polynomials which would figure in the curvature argument is not invariant for $Gl(n, R)$ but only for $SO(n, R)$.

We settle this question for the subclass of affine space forms by an argument analogous to the potential curvature argument.

The point is that all the affine transformations $x \longmapsto Ax + b$ in the discrete group of the space form satisfy: 1 is an eigenvalue of A. For otherwise, the transformation would have a fixed point. But then our theorem is proved by the following

THEOREM. *If a representation* $G \overset{\rho}{\longmapsto} Gl(n, R)$ *satisfies* 1 *is an eigenvalue of* $\rho(g)$ *for all* g *in* G, *then any vector bundle associated by* ρ *to a principal* G*-bundle has Euler characteristic zero.*

COROLLARY. *The Euler characteristic of an affine space form is zero.*

PROOF OF COROLLARY. The tangent bundle of R^n/Γ is associated to the representation

$$\Gamma \xrightarrow{\rho} Gl(n, R),$$

where if $\gamma \in \Gamma$ and $\gamma(x) = Ax + b$ then $\rho(\gamma) = A$.

PROOF OF THEOREM. We can replace G by its real algebraic closure in $Gl(n, R)$ and then pass to the component of the identity without losing our hypothesis which is algebraic or our conclusion because the group of components is finite. Let K be the maximal compact subgroup of this new connected Lie group. We are reduced to a study of cohomology, on the classifying space level of a homomorphism $K \longmapsto SO(n, R)$, which is described by invariant polynomials in the

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corresponding Lie algebras. But the pfaffian which describes the Euler class in cohomology vanishes on the Lie algebra of K because the square of the pfaffian is the determinant and taking logarithms shows every element in the Lie algebra has zero as an eigenvalue. Q.E.D.

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