

COUNTING THE FACES OF CUT-UP SPACES

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Let us take a topological space X and "cut" it along certain subspaces H_1, \dots, H_n . The connected components that remain we call the *regions* of the *topological dissection* of X by the *cut spaces* H_1, \dots, H_n .

How many regions are there? There is a large literature on this question for particular kinds of dissections. For instance much of the study of arrangements of hyperplanes in Euclidean and projective spaces has been concerned with counting regions, bounded regions, and lower-dimensional faces. As another example: into how many pieces is the plane cut by a given set of lines or curves? Answers have been found mainly for straight cuts in low dimensions or general position, some notable exceptions being Steiner's interest in circular and spherical cuts and Winder's and our work on arbitrary arrangements of hyperplanes. (For references see [1, Chapter 18], [2], [5], [6].)

What has been lacking is a unified theory and the power to handle a great variety of dissections. In a forthcoming paper [6] we hope to provide such a theory. Here we shall mention most of the principal results and some areas of application.

1. The fundamental relations. The main idea of our theory is to count regions in terms of the set L of intersections of the cuts or the set L^c of connected components of intersections (we let X be in both sets but \emptyset be in neither). These sets are ordered by inclusion. Thus L (and similarly L^c) has a *Möbius function* (cf. [3]), defined recursively by

$$\mu(S, T) = \delta(S, T) - \sum \{ \mu(S, U) : S \subseteq U \subsetneq T \}, \quad (S, T) \in L^2.$$

Here δ is the Kronecker delta. We also need the *combinatorial Euler number* of a space X , defined by $\kappa(X) = \chi - 1$, where χ means the Euler characteristic of the one-point compactification of X . If X is partitioned into n_k open k -cells ($k = 0, 1, \dots, d$), then $\kappa(X) = n_0 - n_1 + \dots \pm n_d$.

A dissection H of X induces a dissection of $T \in L$ by $H_T = \{H \cap T : \emptyset \neq H \cap T \neq T\}$. The *faces* of H are the regions of all the H_T for $T \in L$.

THEOREM 1. *Suppose X is dissected by H_1, \dots, H_n into regions R_1, \dots, R_m ($m \geq 0$), and every face is a disjoint, finite union of open topological cells.*

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Then

$$(1) \quad \sum_{j=1}^m \kappa(R_j) = \sum_U \mu(U, X) \kappa(U),$$

where U is summed over L or L^c , as desired, and μ is the corresponding Möbius function.

Equation (1) is the key to topological dissection theory. It is proved through Rota's theory of valuations of a distributive lattice [4]. A valuation on D is a function which satisfies $\varphi(s) + \varphi(t) = \varphi(s \vee t) + \varphi(s \wedge t)$. Theorem 1 is a corollary of

THEOREM 2. *If D is a finite distributive lattice with a valuation φ , $P \subseteq D$ contains 0_D and all the join irreducibles of D , and $t \in P$ is not 0_D or join irreducible, then*

$$\sum \{ \mu_P(s, t) \varphi(s) : s \in P, s \leq t \} = 0.$$

2. Properly cellular faces. If, for $T \in L^c$, all the regions of T are open ($\dim T$)-cells, we say the dissection has *properly cellular faces* and denote by f_k the number of k -dimensional faces.

THEOREM 3. *If $\dim X = d$ and X is dissected into properly cellular faces,*

$$(2) \quad \sum_{k=0}^d x^{d-k} f_k = (-1)^d \sum_{T \subseteq U} \mu(U, T) \kappa(U) (-x)^{d-\dim T},$$

where T and U are summed over either L or L^c , as desired.

From (2) follow enumerations of the faces of an arrangement of hyperplanes in Euclidean or projective space, as in [5], or in any convex body; and topological generalizations. If the dissection has general position (any set of k cuts meets in \emptyset or a $(d - k)$ -space), $\mu(U, T) = (-1)^{\dim T - \dim U}$, which greatly simplifies equation (2). Maximizing the number of faces often implies general position. Since also $\mu = \pm 1$ for a polytope dissected by its facets by [4], (2) yields the Dehn-Sommerville equations, and generalizations such as to cubical polytopes.

3. In the sphere. If $X \subseteq S^d$ is dissected by subspaces H_1, \dots, H_n into regions R_1, \dots, R_m , let $A = S^d - (R_1 \cup \dots \cup R_m)$ and let $\beta_k(A)$ be the k th Betti number of A .

THEOREM 4. *Assume that $d \geq 2$; $S^d - X$ and every face of the dissection of X are disjoint, finite unions of open cells; and the regions are open in S^d . Then there are*

$$(3) \quad m = (-1)^d \left(\sum_U \mu(U, X) \kappa(U) - 1 + \beta_0(A) - \dots \pm \beta_{d-2}(A) \right)$$

regions, where U is summed over \mathcal{L} or \mathcal{L}^c , as desired.

The proof is by Theorem 1 and Alexander duality.

In open subsets of S^2 and E^2 , and in P^2 , we can generalize the classical enumerations for arrangements of lines and curves by using (3). For details see [6].

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