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THE INVALIDITY OF THE CALDERON-ZYGMUND INEQUALITY FOR SINGULAR INTEGRALS OVER LOCAL FIELDS

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We will show that the Calderón-Zygmund inequality, $\|T_\omega\|_p \leqslant C(p, r)\|\omega\|_r$, is not valid in the local field setting. A complete proof of the validity of this inequality in the case of singular integrals over $\mathbf{R}^n$ can be found in Dunford and Schwartz, *Linear operators*, Vol. 2. We use the theory of regular functions as developed by M. Taibleson [4] and the F. and M. Riesz theorem for local fields as proved by J. Chao [1].

We assume the reader is familiar with elementary local field analysis and singular integrals in general. In the following work K will denote a local field (nondiscrete, zero-dimensional, locally compact field), $B^n = \{x \in K: |x| \leqslant q^{-n}\}$, $D^n = \{x \in K: |x| = q^{-n}\}$, and ξ_A the characteristic function of the set A. Haar measure λ is normalized so that $\lambda(B^0) = 1$ $(\lambda(B^1) = q^{-1})$ and the prime π is chosen so that $\pi B^0 = B^1$. The fundamental character χ is trivial on B^0 and nontrivial on B^{-1}. C_{00} and C_0 denote the continuous functions with compact support and the continuous functions that vanish at infinity, respectively.

DEFINITION. For $x \in k$, $k \in \mathbf{Z}$, let

$$f(x, -k) = \begin{cases} 0, & k < 2, \\ \xi_{D^0}(x) \sum_{j=2}^{k} \chi(\pi^{-j}x) & \text{if } k \geqslant 2. \end{cases}$$

LEMMA 1. *The function f defined above is regular.*

PROOF. A function g: $K \times \mathbf{Z} \longrightarrow \mathbf{C}$ is said to be regular if

$$g(x, k) = q^{-k} \int_{B^{-k}} g(y - x, k - 1)\, dy.$$

A straightforward calculation shows that f satisfies this equality. $\square$

LEMMA 2.

(a) $$\hat{f}(x, -k) = \frac{q-1}{q} \sum_{j=2}^{k} \xi_{\pi^{-j} + B^0}(x) - \frac{1}{q} \sum_{j=2}^{k} \xi_{\pi^{-j} + D^{-1}}(x).$$

(b) $$\|f(\cdot, -k)\|_2 = \{(q-1)(k-1)/q\}^{1/2} \quad \textit{for } k \geqslant 2.$$

(c) $$\|f(\cdot, -k)\|_r \leqslant \{(q-1)(k-1)/q\}^{(r-1)/r} \quad \textit{for } k \geqslant 2, \quad 2 < r < \infty.$$

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PROOF.

$$\hat{f}(x,-k) = \int_{D^0} f(y,-k)\overline{\chi(xy)}\,dy$$

(a)
$$= \int_{B^0}\sum_{j=2}^{k}\chi((\pi^{-j}-x)y)\,dy - \int_{B^1}\sum_{j=2}^{k}\chi((\pi^{-j}-x)y)\,dy$$

$$= \sum_{j=2}^{k}\xi_{\pi^{-j}+B^0}(x) - \frac{1}{q}\sum_{j=2}^{k}\xi_{\pi^{-j}+B^{-1}}(x)$$

$$= \frac{q-1}{q}\sum_{j=2}^{k}\xi_{\pi^{-j}+B^0}(x) - \frac{1}{q}\sum_{j=2}^{k}\xi_{\pi^{-j}+D^{-1}}(x).$$

(b) $\|f(\cdot,-k)\|_2 = \|\hat{f}(\cdot,-k)\|_2 = \{(q-1)(k-1)/q\}^{1/2}$ by (a).

(c) Let $2 < r < \infty$. As $f(\cdot,-k), \hat{f}(\cdot,-k) \in C_{00}$, $\|f(\cdot,-k)\|_r \leqslant \|\hat{f}(\cdot,-k)\|_{r'}$ where $r' = r/(r-1)$. By (a), $\|\hat{f}(\cdot,-k)\|_{r'} = \{(q-1)(k-1)/q\}^{(r-1)/r}$. □

DEFINITION. For $k \geqslant 2$, let Γ_k denote the set of $\omega: K^* \longrightarrow \mathbf{C}$ such that

(i) $\omega(x + B^k) = \omega(x)$ for $x \in D^0$

(ii) $\omega(\pi^j s) = \omega(x)$ for $x \in K^*, j \in \mathbf{Z}$,

(iii) $\int_{D^0}\omega(x)\,dx = 0$,

and $\Gamma = \bigcup_{k=2}^{\infty}\Gamma_k$.

We note each $\omega \in \Gamma$ is the kernel of a singular integral operator T_ω (see [3]). These kernels correspond to C^∞ kernels in the real case. We denote the multiplier of the operator T_ω by $F(T_\omega)$ and L_p-operator norm of T_ω by $\|T_\omega\|_p$. By $\|\omega\|_r$, we mean $\{\int_{D^0}|\omega(x)|^r\,dx\}^{1/r}$ if $1 \leqslant r < \infty$ and $\sup_{x\in D^0}|\omega(x)|$ if $r = \infty$.

LEMMA 3. *If* $\omega \in \Gamma_k$, *then*

$$F(T_\omega)(y) = \int_{D^0}\omega(x)\sum_{j=1}^{k}\overline{\chi(\pi^{-j}xy)}\,dx \quad \text{for } y \in D^0.$$

Consequently $F(T_\omega)$ *is constant upon the cosets of* B^k *in* D^0.

PROOF. See [2, Proposition 1 and Corollary 2]. □

For the next theorem we preclude the case of even q. The F. and M. Riesz theorem as proved by J. Chao requires this restriction.

THEOREM 1. *For* $1 < p < \infty$, $1 \leqslant r \leqslant \infty$, *there is no constant* $C(p,r)$ *such that for all* $\omega \in \Gamma$,

$$\|T_\omega\|_p \leqslant C(p,r)\|\omega\|_r.$$

PROOF. From the inequalitites $\|T_\omega\|_2 \leqslant \|T_\omega\|_p$ and $\|\omega\|_r \leqslant \|\omega\|_\infty$, we need only show there is no constant C such that $\|T_\omega\|_2 \leqslant C\|\omega\|_\infty$ for all $\omega \in \Gamma$. We will accomplish this if we find a sequence $\{\omega_k\} \subset \Gamma$ such that $\|\omega_k\|_\infty \leqslant 2$ and $\|T_{\omega_k}\|_2 = \|F(T_{\omega_k})\|_\infty \geqslant |F(T_{\omega_k})(1)| \longrightarrow \infty$. To this end we define for $x \in D^0$,

$$g(x,-k)=\begin{cases} f(x,-k)/|f(x,-k)| & \text{if } f(x,-k)\neq 0,\\ 0 & \text{if } f(x,-k)=0,\end{cases}$$

and

$$\omega_k = g(x,-k) - \frac{q}{q-1}\int_{D^0} g(x,-k)\,dx.$$

By the above definition, $\int_{D^0}\omega_k(x)\,dx = 0$ and $\|\omega_k\|_\infty \leqslant 2$. Thus if we extend ω_k to K^* by homogeneity, $\omega_k \in \Gamma_k$. We have

$$\mathcal{F}(T_{\omega_k})(1) = \int_{D^0}\omega_k(x)\sum_{j=1}^{k}\overline{\chi(\pi^{-j}x)}\,dx$$

$$= \int_{D^0}\left|\sum_{j=2}^{k}\chi(\pi^{-j}x)\right| dx + O(1) = \|f(\cdot,-k)\|_1 + O(1).$$

By Lemma 1 the function f is regular. Thus if $\|f(\cdot,-k)\|_1 \leqslant A < \infty$, then f is the regularization of a finite Borel measure μ [4, Theorem 8]. From Lemma 2(a) and the fact that $f(\cdot,-k) \longrightarrow \mu$ in the weak*-topology of the dual of C_{00},

$$\hat{\mu} = \frac{q-1}{q}\sum_{j=2}^{\infty}\xi_{\pi^{-j}+B^0} - \frac{1}{q}\sum_{j=2}^{\infty}\xi_{\pi^{-j}+D^{-1}}.$$

Thus $\hat{\mu}$ is supported on the cone $\Sigma_{j=-\infty}^{\infty}\pi^j(1+B^1)$ and, therefore, μ is absolutely continuous with respect to Haar measure [1, Corollary 5.3]. So μ is given by an L_1-function h and $\hat{h}=\hat{\mu}$. But $\hat{\mu}$ is not in C_0, so h cannot be in L_1. This contradiction gives $|\mathcal{F}(T_\omega)(1)| \longrightarrow \infty$. □

We now give a variation of Theorem 1 by excluding the case $r=\infty$ in (1) and allowing K to be any local field (no restriction on q). The proof is greatly simplified and depends only on Lemma 2.

THEOREM 2. *For* $1<p<\infty$, $1\leqslant r<\infty$, *there is no constant* $C(p,r)$ *such that for all* $\omega\in\Gamma$, $\|T_\omega\|_p \leqslant C(p,r)\|\omega\|_r$.

PROOF. As in the proof of Theorem 1, we may restrict ourselves to $\|T_\omega\|_2$. Let $\beta_k(x)=f(x,-k)$ for $x\in D^0$ and extend β_k to K^* by homogeneity. Observe $\beta_k\in\Gamma_k$. We have

$$\mathcal{F}(T_{\beta_k})(1) = \int_{D^0}\beta_k(x)\sum_{j=1}^{k}\overline{\chi(\pi^{-j}x)}\,dx$$

$$= \int_{D^0}\left|\sum_{j=2}^{k}\chi(\pi^{-j}x)\right|^2 dx + \int_{D^0}\sum_{j=2}^{k}\chi((\pi^{-j}-\pi^{-1})x)\,dx$$

$$= \|f(\cdot,-k)\|_2^2 = (q-1)(k-1)/q$$

by Lemma 2(b). Thus $\|T_{\beta_k}\|_2/\|\beta_k\|_r \longrightarrow \infty$ by Lemma 2(c). For $2\leqslant r<\infty$, this rate of growth is greater than $\{(q-1)(k-1)/q\}^{1/r}$. □

Theorem 2 is valid in the n-dimensional case with no changes except for some constants in the proof. The proof of Theorem 1 can be used except one must find another method to show $\|f(\cdot,-k)\| \longrightarrow \infty$, as the work of Chao is for dimension 1. One would expect $\|f(\cdot, -k)\|_1 = o(\log k)$ as in the case of the classical Dirichlet kernel. A direct computation in the case k is the 2-series field substantiates this conjecture.

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