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AN ITERATIVE SOLUTION OF A VARIATIONAL INEQUALITY FOR CERTAIN MONOTONE OPERATORS IN HILBERT SPACE

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Let A be a multivalued monotone operator on a real Hilbert space H and let C be a nonempty closed convex subset of $D(A)$. If $f \in H$, by a solution of the variational inequality

$$(z_0 - f, x - u_0) \geqslant 0 \quad \forall x \in C, \tag{1}$$

we mean a pair (or, sometimes, just the first component of a pair) $[u_0, z_0] \in A$ satisfying (1) such that $u_0 \in C$. We denote the set of solutions u_0 by E. We shall assume the existence of a solution of (1) and show how to construct it as the weak limit of a sequence $\{x_n\}$ satisfying

$$x_{n+1} = P(x_n - t_n(v_n - f)), \qquad v_n \in Ax_n, \tag{2}$$

where $\{t_n\} \subset [0, \infty)$ and P is the proximity mapping of H onto C. For conditions sufficient to guarantee $E \neq \varnothing$, see Browder **[4]**, Lions **[10]**.

THEOREM 1. *Suppose there exists $u_0 \in E$ such that*

$$\{(v - f, x - u_0) = 0, x \in C, v \in Ax\} \Rightarrow x \in E. \tag{3}$$

If, in (2), $\Sigma t_n = \infty$, $\Sigma \|t_n(v_n - f)\|^2 < \infty$, *and* $\{v_n\}$ *is bounded, then* $\{x_n\}$ *converges weakly to a point of E.*

Note, in particular, that if A is bounded on C, then for any nonnegative sequence $\{t_n\}$ in $l^2 \setminus l^1$ the conditions on $\{t_n\}$ and $\{v_n\}$ are automatically satisfied.

THEOREM 2. *If A has the property*

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(4) $\{z_1 \in Ax_1, z_2 \in Ax_2, (z_1 - z_2, x_1 - x_2) = 0\} \Rightarrow z_2 \in Ax_1,$

then A *satisfies* (3) *on any* C *for which* $E \neq \emptyset$.

Condition (4) is satisfied by a very wide class of monotone operators which have arisen in several different contexts: strictly monotone operators, subdifferentials of proper l.s.c. convex functions, the maximal monotone trimonotone operators of Brezis and Browder [1], [2], maximal monotone operators satisfying condition (1) of [1], and the class M_2 of Browder and Petryshn [5] (single-valued operators satisfying $(Ax - Ay, x - y) \geqslant \delta \|Ax - Ay\|^2$ for some $\delta > 0$). In particular, (2) can be applied to minimize a convex functional on a constraint set under much weaker hypotheses than has heretofore been possible (compare with Goldstein [9]).

Moreover, in an important special case (3) is satisfied by an otherwise arbitrary monotone operator.

THEOREM 3. *If* int $C \neq \emptyset$, C *is rotund, and* (1) *has a solution* $[u_0, z_0]$ *with* $z_0 \neq f$, *then* A *satisfies* (3) *and* u_0 *is the unique solution of* (1). *If, in addition,* C *is uniformly rotund, then in Theorem* 1 *the hypothesis that* $\{v_n\}$ *is bounded may be deleted and the conclusion strengthened to*: $\{x_n\}$ *converges* **strongly** *to the solution of* (1).

PROOFS. In what follows, we shall normalize to $f = 0$. For any solution $[u, z]$ of (1) we find

$$(5)\qquad \begin{aligned} 0 &\leqslant 2t_n(z, x_n - u) \leqslant 2t_n(v_n, x_n - u) \\ &\leqslant \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + \|t_n v_n\|^2 \end{aligned}$$

by virtue of the nonexpansiveness of P, $Pu = u$, the monotonicity of A, and the fact that $[u, z]$ is a solution of (1). Inequality (5) permits three conclusions: (a) $\lim_n \|x_n - u\|$ exists for each $u \in E$; (b) $\Sigma\, t_n(v_n, x_n - u_0)$ is a convergent positive-term series, hence $\liminf_n (v_n, x_n - u_0) = 0$; (c) if a subsequence $\{x_{n(i)}\}$ converges weakly to x and $\lim_i (v_{n(i)}, x_{n(i)} - u_0) = 0$, then $x \in E$ (because $0 \leqslant (z_0, x_n - u_0) \leqslant (v_n, x_n - u_0)$ and (3) is satisfied). An appeal to a variant of [3, Lemma 6] (or a direct appeal to Opial's lemma [11]) establishes the uniqueness of x in (c), i.e.,

$$(6)\qquad \begin{aligned} &\exists\, x^* \in E \text{ such that } x_{n(i)} \rightharpoonup x^* \,\forall \{x_{n(i)}\} \\ &\text{satisfying } \lim_i (v_{n(i)}, x_{n(i)} - u_0) = 0. \end{aligned}$$

For $\delta > 0$ put $P = \{n\colon (v_n, x_n - u_0) \geqslant \delta\}$ and note that (5), the nonexpansiveness of P, and the boundedness of $\{v_n\}$ imply that $\Sigma_{n \in P} \|x_n - x_{n+1}\|$ converges. With (6) this implies $x_n \rightharpoonup x^*$.

The proof of Theorem 2 is simple computation which we omit. The proof of Theorem 3 is based on the observation that if $[u_0, z_0]$ is a solution of (1) with $z_0 \neq 0$, then necessarily $u_0 \in \text{bdry}\, C$ and the hyperplane $u_0 + z_0^\perp$ supports C at u_0. Any solution $[x, v] \in A$, $x \in C$ of $(v, x - u_0) = 0$ must have $x \in u_0 + z_0^\perp$ because $0 = (v, x - u_0) \geqslant (z_0, x - u_0) \geqslant 0$; by the rotundity of C, therefore, $x = u_0$, i.e., (3) is satisfied. This implies, incidentally, that the solution u_0 of (1) is unique (although z_0 may not be unique).

If C is also uniformly rotund, then, as in the proof of Theorem 1, there exists $x^* \in E$ for which (6) is valid (and such subsequences $\{x_{n(i)}\}$ do exist). But a sequence in a uniformly rotund convex set which converges weakly to a point on the boundary must converge strongly; since $\lim_n \|x_n - x^*\|$ exists, therefore $\lim_n \|x_n - x^*\| = \lim_i \|x_{n(i)} - x^*\| = 0$.

For related iterative solutions of $f \in x + Ax$ and $f \in Ax$ (without use of the projection P) see [7], [8]. The proof of Theorem 1 is based on an idea of [6]. Complete proofs, other consequences, extensions of the results of [8] to variational inequalities, as well as sequential analogue of [6, Theorem 5] for even convex functions, will appear elsewhere.

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