and Marcel-P. Schützenberger published *Théorie géométrique des polynômes Eulériens*, Springer-Verlag Lecture Notes in Mathematics, no. 138, 1970. There is considerable overlap between the contents of the Foata–Schützenberger lecture notes and these Foata lecture notes.

What is Foata's contribution to the development of the foundations of enumerative combinatorics? After developing the elementary properties of exponential generating functions and the theory of formal power series, Foata discusses nonabelian and abelian partitional compositions. (Foata and Schützenberger introduced the partitional composition notation in their 1970 lecture notes, cited above.) A classical example of such a composition is a permutation of an n-set expressed as the abelian partitional composition of disjoint cycles. Foata obtains many classical enumerative results from what he calls the fundamental transformation. This transformation is a bijection of the symmetric group S_n onto itself such that a permutation with k exceedances is mapped to a permutation with k descents.

What are some of the results that can be obtained from partitional compositions, the fundamental transformation, and exponential generating functions? Foata obtains generating functions for Eulerian polynomials and for Laguerre polynomials. He finds generating functions for permutations enumerated by the number of cycles and for permutations without fixed points. The fundamental transformation defined for permutations of an n-set is extended to all mappings of an n-set into itself. Finally, Foata obtains generating functions for acyclic functions enumerated by the number of fixed points, by the number of elements in each orbit, by height, and by the number of inversions.

Now, let us consider the chapter written by Bernard Kittel. He shows that six known probabilistic identities can be proven by an exponential generating function formula. It is exciting to see this interplay between these two areas of mathematics, probability and combinatorics. Kittel is applying combinatorial techniques to probability theory, in contrast to the usual application of probabilistic techniques to combinatorics.

Clearly, these lecture notes should be read by those studying the foundations of combinatorial enumeration. These notes seem too specialized to be of great interest to the mathematical community at large. This community needs a survey book which puts the various approaches to the foundations of enumeration in proper perspective. Such approaches include combinatorial mappings, combinatorial operators, generating functions, incidence algebras, and umbral (Blissard) calculus. At present such a book is not available.

EARL GLEN WHITEHEAD, JR.

Theory of branching of solutions of non-linear equations, by M. M. Vainberg and V. A. Trenogin, Monographs and Textbooks on Pure and Applied Mathematics, Noordhoff International Publishing, Leyden, 1974, xxvi+485 pp.

This book is a translation of the original Russian work, published in 1969.

Bifurcation theory is the analysis of branch points of nonlinear functional equations in a vector space, usually a Banach space. Thus, suppose $G(x, y)$ is a smooth (Fréchet differentiable) mapping from $\mathscr{E}\times\mathscr{F}$ to $\mathscr{G}$, where $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ are Banach spaces, that $G(0, 0)=0$, and that $G_x(0, 0)$, the Fréchet derivative of G relative to x, is a linear transformation mapping $\mathscr{E}$ to $\mathscr{G}$. One is then interested in constructing all solutions of the functional equation

$$G(x, y)=0 \tag{$*$}$$

in a neighborhood of $x = 0$, $y = 0$, say, under the assumption that $G_x(0, 0)$ has a finite dimensional null space and a closed range of finite codimension. The subject is of fundamental importance for applied mathematics, as it is naturally of concern in any physical problem described by a nonlinear system of equations which depends on some set of parameters. Equation $(*)$ may in practice be taken to represent a nonlinear system of partial differential equations, integral equations, ordinary differential equations, etc. Usually y is taken to be a real or complex parameter or finite set of parameters. The fact that $G_x(0, 0)$ has a nontrivial null space often reflects the physical fact that $y=0$ is a critical parameter value for the system under study.

Bifurcation theory is thus a broad subject; and, moreover, it is one which is currently in a very rapid phase of development. It touches on many areas of pure mathematics, especially functional analysis and differential equations. With this overview of the subject in mind, let me discuss the book of Vainberg and Trenogin and try to place its contribution as a textbook in the current literature on bifurcation theory.

I especially like the first two chapters of the book. In these two chapters the authors deal in some depth with the branching of solutions of equations in the finite dimensional case. This is often the heart of the issue since in a large number of applications the infinite dimensional problem $(*)$ can be reduced to the finite dimensional case by the now-standard "Lyapounov-Schmidt" procedure. The branching equations, or bifurcation equations, consist in general of m equations in n unknowns, where m is the codimension of the range of $G_x(0, 0)$ and n is the dimension of the null space.

In Chapter 1 the one-dimensional branching equation ($n=m=1$) is discussed; and a thorough account of the method known as "Newton's polygon" or "Newton's diagram" is given, along with a number of useful illustrative examples. In Chapter 2 the multidimensional branching problem is discussed. The authors give a resumé of divisibility theory and its application to the branching problem.

(Incidentally, with regard to the material in the first two chapters, the interested mathematician should also consult the survey article by D. Sather, *Branching of solutions of nonlinear equations*, Rocky Mountain J. Math. **3** (1973), 203–250.)

In Chapters 3, 4, and 5 the authors give an extensive account of a class of nonlinear integral equations. The detail with which the computations are worked out here is certainly admirable. However, the material here is a specific case of the general theory, which is carried out more generally and more concisely in Chapter 7 under the heading "Nonlinear equations in Banach spaces". The student first learning the subject might find the general presentation of Chapter 7 more understandable on a first reading.

In Chapter 6 the branching theory of periodic solutions of ordinary differential equations is discussed. I think most of this material is already widely available in books on ordinary differential equations (for example, Coddington and Levinson's *Theory of ordinary differential equations*, McGraw-Hill, 1955; and especially J. K. Hale's book, *Ordinary differential equations*, Wiley-Interscience, 1969, which is not referred to). The authors consider both autonomous and nonautonomous equations. However, no discussion is given to the important 1942 work of E. Hopf on the bifurcation of periodic solutions from equilibrium solutions, nor (quite understandably) to the more recent advances in this subject since 1970 which have extended Hopf's work to systems of partial differential equations. (The book first appeared in 1969.)

Nonlinear functional analysis is the topic of Chapter 7. The implicit function theorem in a Banach space is stated without proof. The analytic version is stated and its proof by a majorant method is sketched. (The analytic version follows from the C^1 version if one observes that a mapping from one complex Banach space to another is analytic if it is differentiable.) The Lyapounov-Schmidt method is discussed for nonlinear functional equations and the relationship of the solutions of the branching equations and the original problem is worked out.

Chapter 8 contains a discussion of the branching problem when the operator $G_x(0, 0)$ is not Fredholm but nevertheless has a finite dimensional null space and a closed range of finite codimension. In this case the index $\chi = n - m$, where $n = \dim \mathcal{N}(G_x(0, 0))$ and $m = \operatorname{codim} \mathcal{R}(G_x(0, 0))$, is nonzero. Applications to nonlinear singular integral equations are discussed. These equations involve operators of the type

$$By = a(s)y(s) + P\frac{1}{\pi i}\int_L \frac{K(s, t)y(t)}{t-s}\,dt$$

where s lies on the contour L and the Cauchy principle value of the integral is understood, L being a smooth closed contour in the complex plane. The index χ of the operator B is given by

$$\chi = n - m = \frac{1}{2\pi}\left[\arg\frac{a(s) - K(s, s)}{a(x) + K(s, s)}\right]_L$$

where $[\quad]_L$ denotes the increment of the enclosed function over a circuit of the contour L.

In the same chapter nonlinear elliptic boundary value problems are discussed, the boundary condition being $\alpha\, du/dl + \beta u = \varphi$ on the boundary Γ, where l is a continuously turning direction on Γ. Boundary value problems in which l is tangent to Γ at some points of Γ may lead to problems where the index $\chi \neq 0$, but such problems are omitted from the discussion.

Other topics discussed in the book are Jordan chains (or Jordan bases) for

Fredholm operators, linear branching theory for eigenfunctions and eigenvalues, and bifurcation from infinity (that is, solutions of $G(x, \lambda)=0$ which tend to infinity as $\lambda \to 0$).

In Chapter 10 the authors give applications of the theory to a number of applied problems, including the buckling of elastic rods and plates, standing waves on a fluid surface, and oscillations of an orbiting satellite.

Vainberg and Trenogin's book should prove to be a valuable reference for bifurcation theory in years to come. The book contains some rather good accounts of topics which do not currently enjoy the attention they deserve, as well as an extremely useful bibliography. On the other hand, the reader must be aware of the fact that a large body of important results in the subject has come into existence in the six years since the book was written. The book contains no discussion of the relationship between bifurcation and loss or exchange of stability, no topological degree theory, no variational theory, and no discussion of the bifurcation of periodic motions from an equilibrium solution as it loses stability. The student interested in using the book as an introductory text will probably find the material in Chapters 1, 2, 6, and 7 to be the most important.

DAVID H. SATTINGER