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C^*-ALGEBRAS GENERATED BY COMMUTING ISOMETRIES. I[1]

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We announce results on structure and Fredholm theory for the C^*-algebra $A(V, W)$ generated by two isometries V and W on a separable Hilbert space with the properties (where $[A, B] \equiv AB - BA$) that $[V, W] = 0$ and $[V, W^*]$ is compact. Such algebras arise naturally in several contexts: first, in the stuc, of singular integral operators with generally discontinuous symbols; secondly, in the study of difference operators on certain domains in R^2. Proofs of the results described here and extensions to the case of more than two generators will appear elsewhere [1].

Let $L^2(T)$ denote the space of Lebesgue square-integrable complex-valued functions on the circle T and let $H^2(T)$ be the Hardy subspace of $L^2(T)$ with P_+ the orthogonal projection from $L^2(T)$ onto $H^2(T)$. Now for Φ in $L^\infty(T)$, we define the Toeplitz operator T_Φ by $T_\Phi f = P_+ \Phi f$. We recall than an inner function is an element of $H^2(T)$ which has modulus one a.e. The only continuous inner functions are the finite Blaschke products and it is a property of inner functions that their discontinuities are of the rapidly oscillating type rather than jumps.

Our analysis is based on a careful examination of the product of generators, VW. There is a reducing space $H = \{x: x = (VW)^n y_n \forall n > 0\}$ for VW so that $VW|_H$ is unitary and since $[V, W] = 0$, H reduces V and W with $V|_H$, $W|_H$ both unitary. Moreover, there is a unitary equivalence between $H^\perp$ and $H^2(T) \otimes l_2(S)$ where S is a subset of the integers Z so that $VW|_{H^\perp}$ becomes $T_z \otimes I$. Careful analysis now shows that under the same unitary equivalence, we have

$$V|_{H^\perp} \approx T_z \otimes UP + I \otimes U(I - P), \qquad W|_{H^\perp} \approx T_z \otimes (I - P)U^* + I \otimes PU^*,$$

where U is some unitary and P is some orthogonal projection operator on $l_2(S)$. In fact, for arbitrary U, P one obtains commuting isometries by the above formula and the pair (U, P) is easily seen to be a complete unitary invariant for the pair $(V|_{H^\perp}, W|_{H^\perp})$. It is not hard to see that the analysis above

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extends to the case of n commuting isometries.

We now have an obvious unitarily implemented embedding

$$\rho: \mathcal{A}(V, W) \longrightarrow \mathcal{A}(V|_H, W|_H) \oplus [\mathcal{A}(T_z) \otimes \mathcal{A}(U, P)].$$

We recall that for A in the C^*-algebra $\mathcal{A}$ acting on the Hilbert space $\widetilde{H}$, A is Fredholm if and only if $[A]$ is invertible in $\mathcal{A}/\mathcal{A} \cap \mathcal{K}$ where $\mathcal{K}$ is the ideal of compact operators on $\widetilde{H}$. The C^*-algebra $\mathcal{A}(V|_H, W|_H)$ is commutative so the index of any Fredholm operator in this algebra must be zero (in general, index A = dim ker A − dim ker A^*). Hence, the interesting part of the Fredholm theory of $\mathcal{A}(V, W)$ reduces to an analysis of $\mathcal{A}(T_z) \otimes \mathcal{A}(U, P)$. Techniques are available to deal with Fredholm theory in such tensor products but it is first necessary to analyze the pair (U, P) and the algebras $\mathcal{A}(U, P)$ in some detail.

It is easy to check that $\rho([V, W^*]) = 0 \oplus \{-P_0 \otimes UPU(I - P)\}$ where P_0 is the projection onto the constants in $H^2(T)$, so that $[V, W^*]$ is compact if and only if $PU(I - P)$ is compact. It follows easily that $(PUP)(PU^*P) = P +$ compact, so there are three possible cases: Case 1, $PUP|_{Pl_2(S)}$ is Fredholm of index zero; Case 2, $PUP|_{Pl_2(S)}$ is Fredholm of index other than zero; and Case 3, $PUP|_{Pl_2(S)}$ is semi-Fredholm with index $+\infty$.

THEOREM. *In Case* 1, *there is a unitary R so that $U' = R^*UR$ is a compact perturbation of a diagonal operator* (*an operator having the form $Af_n = \lambda_n f_n$ for some orthonormal basis $\{f_n\}$*) *and $P' = R^*PR$ is diagonal in the same basis. In Case* 2, *there is a unitary R so that $U' = R^*UR$ is a compact perturbation of some power of the bilateral shift B or its adjoint on $l_2(Z)$* (*$Be_n = e_{n+1}$ for the canonical basis*) *and $\widetilde{P}_+ = R^*PR$ is the projection from $l_2(Z)$ onto $l_2(Z_+)$ where Z_+ is the set of nonnegative integers. In Case* 3, *there is a unitary R mapping $l_2(Z) \otimes l_2(Z)$ onto $l_2(S)$ so that $U' = R^*UR$ is a compact perturbation of $B^* \otimes I$ and $R^*PR = \widetilde{P}_+ \otimes I$.*

Using this Theorem, we can give explicit characterizations of Fredholm operators in $\mathcal{A}(V, W)$ and formulas for the index of such operators are obtained. Exact statements and the proofs of these results are technical and are left to [1].

We now wish to discuss the motivating examples for this development. It is easy to check that if either V or W has finite defect then P or $I - P$ has finite rank so that Case 1 occurs. In particular, if φ is an arbitrary discontinuous inner function then $\mathcal{A}(T_z, T_\varphi)$ is in Case 1. Such algebras arise in the study of singular integral operators with (possibly) discontinuous symbols on the circle. The discontinuities here are of the rapidly oscillating type rather than the jump discontinuities considered by other authors. In fact, if φ_1 and φ_2 are inner functions such that $[T_{\varphi_1}, T^*_{\varphi_2}]$ is compact then $(T_{\varphi_1}, T_{\varphi_2})$ is in Case 1 and we have been able to determine the Fredholm theory for operators in $\mathcal{A}(T_{\varphi_1}, T_{\varphi_2})$ in a "natural" topological manner.

The other motivating example for this study comes from consideration of certain finite-defect bi-invariant subspaces $\widetilde{H}$ of the Hardy space of the bidisc $H^2(T^2)$ and the algebras $\mathcal{A}(T_z|_{\widetilde{H}}, T_w|_{\widetilde{H}})$ generated by multiplications by z and w restricted to $\widetilde{H}$. In the special case that $\widetilde{H}$ is the space of all inverse Fourier transforms of $l_2(\widetilde{S})$ for $\widetilde{S}$ a subset of $Z_+ \times Z_+$, it is easy to see that $\mathcal{A}(T_z|_{\widetilde{H}}, T_w|_{\widetilde{H}})$ is unitarily equivalent via the Fourier transform to the algebra of all difference operators on $l_2(\widetilde{S})$. For such $\widetilde{H}$ and the pair $(T_z|_{\widetilde{H}}, T_w|_{\widetilde{H}})$ we can check that $U = B^*$ and P is a finite-rank perturbation of $\widetilde{P}_+$ on $l_2(Z)$ so that $PUP|_{Pl_2(Z)}$ has index one and $(T_z|_{\widetilde{H}}, T_w|_{\widetilde{H}})$ is in Case 2. The C^*-algebras $\mathcal{A}(T_z|_{\widetilde{H}}, T_w|_{\widetilde{H}})$ for all of these special $\widetilde{H}$ turn out to be unitarily equivalent, a fact which is of independent interest.

Finally, we remark that our method extends to the study of strongly continuous one-parameter semigroups of isometries. In the purely nonunitary case, we obtain a representation of generators $V_t \approx T_z \otimes U_tP_t + I \otimes U_t(I - P_t)$, $1 \geqslant t \geqslant 0$. There is a "covering" relation between the usual unitary extension of $\{V_t\}$ and $\{U_t\}$ and we find that, up to the multiplicity of the semigroup, U_t is rotation by $2\pi t$ on $L^2(T)$ and P_t is multiplication by the characteristic function of $\{e^{i\theta}: 2\pi(1-t) \leqslant \theta \leqslant 2\pi\}$. In the case that $V_t = T_{\varphi_t}$ on $H^2(T)$, where

$$\varphi_t(z) = \exp\left\{-t\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\right\}$$

for μ some singular probability measure on T, we can analyze the structure of $\mathcal{A}(T_z, T_{\varphi_t}, t \geqslant 0)$. We find that in our representation

$$T_z \approx T_{z - e^{-1}/(1 - e^{-1}z)} \otimes UP + I \otimes U(I - P).$$

Moreover, $[U, U_t] \equiv 0$ and we can unitarily embed $\mathcal{A}(T_z, T_{\varphi_t}, t \geqslant 0)$ in a suitable product $\mathcal{A}(T_z) \otimes \mathcal{A}(U, P, U_t, P_t, t \geqslant 0)$ to determine a natural "symbol space" and index theorem.

REFERENCE

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