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DEPARTMENT OF MATHEMATICS, CARNEGIE-MELLON UNIVERSITY, PITTSBURGH, PENNSYLVANIA 15213

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MAXIMA IN BROWNIAN EXCURSIONS

BY KAI LAI CHUNG[1]

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Let $\{X(t), t \geq 0\}$ be the standard one-dimensional Brownian motion starting at 0. For $t > 0$ define

$$T(t) = \sup\{s \leq t | X(s) = 0\}; \qquad T'(t) = \inf\{s \geq t | X(s) = 0\};$$

$$L^-(t) = t - T(t); \qquad L(t) = T'(t) - T(t);$$

$$M^-(t) = \max_{T(t) \leq s \leq t} |X(s)|; \qquad M(t) = \max_{T(t) \leq s \leq T'(t)} |X(s)|.$$

The random time interval $(T(t), T'(t))$ is the *excursion interval straddling* t,

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and X restricted to this interval is the corresponding *excursion process.* Paul Lévy initiated the "profound study" of these excursions; see [5, Chapter 6]. An expanded account is found in Itô-McKean [4]. It turns out that X restricted to the interval $(T(t), t)$ is also interesting by itself, and following D. Iglehart, we shall call it the *meandering process ending at t.* Iglehart suggested the investigation of the maximum of the latter process denoted by $M^-(t)$ above. It will be seen that the conditional distribution of $M^-(t)$ given the duration $L^-(t)$ of the meandering is the same for all t greater than the given value of the duration. The conditional distribution of $M(t)$ given $L(t)$ depends neither on t nor on $T(t)$. The latter is a consequence of Lévy's results, but the explicit determination of the various distributions given below seems to be new, and more can be done in this direction.

Our method is based on an analysis of the last exit from zero occurring at $T(t)$. There is a far-reaching analogy between this and the last exit from an arbitrary fixed state in a continuous parameter Markov chain. The facts pertaining to the latter case are expounded in [1, §II.12] and carried on under the guise of a boundary state in [2]. Indeed, the abundant formulas in the Brownian case tend to obscure by their explicitness, lending to unperceived cancellations and juxtapositions. The ideas are clearer for chains, and transferring them to Brownian motion is a pleasant and rewarding experience. On the other hand, the problem of the maxima is peculiar to the well ordering of the reals and furnishes a testing ground for the general methodology.

Let us put

$$p(t; x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(x-y)^2}{2t}\right\},$$

$$\bar{p}(t; x, y) = p(t; x, y) - p(t; -x, y),$$

$$g(t; 0, x) = \frac{|x|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{x^2}{2t}\right\} = f(t; x, 0).$$

Here then is the fundamental "last exit decomposition":

$$p(t; 0, y) = \int_0^t p(t-s; 0, 0) g(s; 0, y)\, ds; \tag{1}$$

for $0 < s < t$. As an identity in calculus, this coincides with the "first entrance decomposition":

$$p(t; 0, y) = \int_0^t f(s; 0, y) p(t-s; y, y)\, ds. \tag{2}$$

Thus the entrance law (g_t) to the taboo semigroup $(\bar{p}_t)$, which corresponds to the difficult (g_{ij}) in Markov chains (see [1, Theorem 3, §II.12]), is identical with the exit law (f_t) which corresponds to the easy (f_{ij}) (Theorem 5, loc. cit.). But it is the probabilistic meaning of (1) that counts, and this is rendered as follows:

$$P\{X(t) \in dy\} = \int_{s=0}^{t} P\{T(t) \in ds;\, X(t) \in dy\}. \tag{3}$$

Cf. Theorem 7, loc. cit. Now (3) has a vital extension which will be shown in the simplest case. For $0 < s < s + \delta < t$, we have, writing henceforth $Y(t)$ for $|X(t)|$:

$$\begin{aligned} &P\{T(t) \in ds;\, Y(s+\delta) \in dx;\, Y(t) \in dy\} \\ &\qquad = p(s; 0, 0)ds\, g(\delta; 0, x)dx\, \bar{p}(t-s-\delta; x, y)dy. \end{aligned} \tag{4}$$

With the factor $g(\delta; 0, x)$ above we have made the entrance into the meandering and excursion processes, and the path is set. It turns out that conditioning with respect to $L^-(t)$ will be more appropriate and we obtain for $0 < \delta < r$:

$$\begin{aligned} &P\{Y(\delta) \in dx;\, Y(r) \in dy \mid L^-(t) = r\} \\ &\qquad = \sqrt{2\pi r}\, g(\delta; 0, x)dx\ \bar{p}(r-\delta; x, y)dy. \end{aligned} \tag{5}$$

The factor $\sqrt{2\pi r}$ is one of those obscurities alluded to above and is really the reciprocal of $\int_0^\infty g(r; 0, x)\, dx$.

The method described above gives quick and standardized derivations of results such as those below.

THEOREM 1. *For $\xi > 0, y > 0, 0 < r < t$, we have*

$$\begin{aligned} &P\{M^-(t) \leqslant \xi;\, Y(t) \in dy \mid L^-(t) = r\} \\ &\qquad = \sqrt{2\pi r}\left\{\sum_{n=0}^{\infty} g(r; 0, 2n\xi + y) - \sum_{n=1}^{\infty} g(r; 0, 2n\xi - y)\right\}. \end{aligned} \tag{6}$$

Integrating out dy, we obtain

$$P\{M^-(t) \leqslant \xi \mid L^-(t) = r\} = 1 + 2\sum_{n=1}^{\infty} (-1)^n \exp\left\{-\frac{n^2\xi^2}{2r}\right\}. \tag{7}$$

It follows from (6) and (1) that

$$P\{M^-(t) \leqslant \xi,\, Y(t) \in dy\} = 2p(t; 0, y) - 2\sum_{n=1}^{\infty} \bar{p}(t; 2n\xi, y), \tag{8}$$

$$P\{M^-(t) \leqslant \xi\} = 2 \sum_{n=0}^{\infty} (-1)^n P\{n\xi \leqslant X(t) < (n+1)\xi\}. \tag{9}$$

John B. Walsh derived (9) directly by the method of reflections.

THEOREM 2. *For $\xi > 0$, $0 < r < \min(t, u)$, we have*

$$P\{M(t) \leqslant \xi | L(t) = u, L^-(t) = r\}$$

$$= 1 + 2 \sum_{n=1}^{\infty} \left(1 - \frac{(2n\xi)^2}{u}\right) \exp\left\{-\frac{(2n\xi)^2}{2u}\right\}.$$

It is not obvious that the last expression represents a distribution function in ξ for $0 < \xi < \infty$, as vouchsafed by the theorem. This can be verified via Laplace transforms and Euler's great expansion of $(e^z - 1)^{-1}$. Another independent check was given by W. A. Veech who used the functional equation of the theta function.

W. D. Kaigh informed me that he had derived the distributions above by finding the appropriate limiting theorems for sums of independent and identically distributed symmetric Bernoullian random variables conditioned on "$T > n$" or "$T = n$", where T is the first time for the sum to be zero. Such a scheme was considered by Iglehart [3] who proved an analogue of the classical invariance principle for it.

A fuller account of the results mentioned here will be published elsewhere.

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DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305