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LIMITS OF SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS[1]

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Consider the Volterra integral equation

$$\text{(E)} \qquad u(t) = -\int_0^t A(t-\tau)g(u(\tau))\,d\tau + f(t), \qquad t > 0,$$

on a Hilbert space H. $A(t)$ is a family of bounded, linear, selfadjoint operators on H and g is a nonlinear bounded map from H into itself. If $f(t) \longrightarrow f_0(t)$ as $t \longrightarrow \infty$ then

$$(\text{E}_0) \qquad u_0(t) = -\int_0^\infty A(\tau)g(u_0(t-\tau))\,d\tau + f_0(t), \qquad t > 0,$$

will be called a *limit* equation for (E). The following result appears in [7].

THEOREM (MILLER). *Let $H = R^n$. Suppose $A \in L_1(0, \infty)$, $f: R^+ \longrightarrow R^n$ is bounded and uniformly continuous, g is continuous. Let* (E) *have a bounded solution u on R^+. Then there exist a solution u_0 of* (E_0) *and a sequence $t_n \longrightarrow \infty$ such that $u(t + t_n) \longrightarrow u_0(t)$ as $n \longrightarrow \infty$.*

We give a result complementary to Miller's. We give conditions on A and g which guarantee that if (E_0) has a bounded solution then all solutions of (E) tend to u_0 as $t \longrightarrow \infty$.

Our hypotheses are taken from [5]. We assume that g is continuous, bounded with $g(0) = 0$ and that

$$(1) \qquad \langle g(u) - g(v), u - v\rangle \geqslant m\|u - v\|^2 \quad \text{for some } m > 0.$$

We assume that $A \in C^{(2)}[0, \infty)$, $A^{(k)} \in L_1(0, \infty)$, $k = 0, 1, 2$. A also is to satisfy

$$(2) \quad \langle A(0)u, u\rangle \geqslant \alpha\|u\|^2, \quad \langle \dot{A}(0)u, u\rangle \leqslant -\beta\|u\|^2, \qquad \alpha > 0, \quad \beta > 0,$$

$$(3) \qquad \begin{gathered}\text{given any } N, \text{ there exists } \delta(N) > 0 \text{ such that} \\ \langle \operatorname{Re} \hat{A}(i\eta)u, u\rangle \geqslant \delta(N)\|u\|^2 \quad \text{for all } |\eta| \leqslant N.\end{gathered}$$

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In (3) $A^\wedge(s)$ is the Laplace transform of A. (See [6] for a comparison of (3) with the monotonicity and convexity conditions of [4].) f is to satisfy

(4) $$f(t) = f_0(t) + h(t) \quad \text{where } h \in L_1(0, \infty) \cap L_2(0, \infty).$$

THEOREM (1). *Suppose* (E_0) *has a solution which is bounded on* R^+. *Then any solution of* (E) *satisfies* $u(t) - u_0(t) \longrightarrow 0$ *as* $t \longrightarrow \infty$.

The proof involves ideas from [5]. (E) can be solved for $g(u)$. (E_0) can be written in the form (E) with a forcing term depending on u_0 and the resulting equation solved for $g(u_0)$. One subtracts the resulting equations, multiplies by $u - u_0$ and integrates from 0 to T. Conditions (1)–(4) then yield an energy estimate which shows that $u - u_0 \in L_\infty(0, \infty) \cap L_2(0, \infty)$. (E) and ($E_0$) can then be used to establish that $u - u_0$ is uniformly continuous, hence $u - u_0 \longrightarrow 0$ as $t \longrightarrow \infty$.

There are two immediate applications.

THEOREM (2). *Let the hypotheses of Theorem* (1) *hold and suppose* $f_0(t)$ *in* (4) *equals* f_0 *a constant. Then* (E_0) *has a constant solution* u_0 *and all solutions of* (E) *tend to* u_0 as $t \longrightarrow \infty$.

PROOF. We need only show that (E_0) has a constant solution. From (3) it follows that $A = A^\wedge(0)$ has a positive selfadjoint square root. The equation to be solved can be reduced to

(5) $$v_0 + A^{1/2} g(A^{1/2} v_0) = v_0 + F(v_0) = A^{-1/2} f_0.$$

We have

$$\langle F(v_0) - F(v_0'), v_0 - v_0' \rangle = \langle g(A^{1/2} v_0) - g(A^{1/2} v_0'), A^{1/2} v_0 - A^{1/2} v_0' \rangle$$
$$\geqslant m \| A^{1/2}(v_0 - v_0')\|^2 > 0.$$

Hence F is a continuous monotone operator and it follows by a result of Minty [8] that (5) has a solution.

The second application concerns the existence of periodic limits. Let P_T denote the set of all T-periodic functions u on $L_2((-\pi, \pi) : H)$. For $u \in P_T$ we have a Fourier series $u = \Sigma_{-\infty}^{+\infty} u_n e^{int}$. Consider the linear operator $\mathcal{A}$ defined by $\mathcal{A} v(t) = \int_0^\infty A(\tau) v(t - \tau)\, d\tau$. For $v \in P_T$,

(6) $$\mathcal{A} v = \Sigma A^\wedge(in) v_n e^{int}.$$

it is shown in [5] that the hypotheses on A imply that $\|A^\wedge(s)\| = O(s^{-1})$

as $s \longrightarrow \infty$. From this one concludes that A is a bounded linear map from P_T into $P_T \cap L_\infty((-\pi, \pi) : H)$. If H is finite dimensional the map from P_T into itself is compact. The same remarks hold for the nonlinear map $v \longrightarrow \int_0^\infty A(\tau)g(v(t-\tau))d\tau$ if g satisfies

$$\|g(v)\| \leqslant K\|v\|. \tag{7}$$

Assume now that $f_0 \in P_T \cap L_\infty((-\pi, \pi) : H)$ and the hypotheses of Theorem (1) hold.

THEOREM (3). *Suppose $g(u) = Lu$ where L is a bounded linear selfadjoint operator. Then* (E_0) *has a solution $u_0 \in P_T \cap L_\infty((-\pi, \pi) : H)$ and all solutions of* (E) *tend to* (E_0) *as $t \longrightarrow \infty$.*

PROOF. A solution can be found in the form $u_0 = \Sigma u_n^0 e^{int}$ where $(I + A^\wedge(in)L)u_n^0 = f_n^0, f(t) = \Sigma f_n^0 e^{int}$. The hypotheses on A and L guarantee that for each n, $(I + A^\wedge(in)L)^{-1}$ exists as a bounded operator with $\|(I + A^\wedge(in))^{-1}\| \leqslant J$ (see [5]). It follows easily that the series yields the desired solution. If f_0 is continuous then u_0 is continuous.

Theorem (3) can be used with a perturbation argument and the contractive mapping principle to yield the following result.

THEOREM (4). *Let $g(u) = Lu + \epsilon h(u)$, where L is as in Theorem* (3), *$h(0) = 0$, h satisfies* (1) *and h is globally Lipschitz. Then for ϵ sufficiently small* (E_0) *has a solution u_0 and all solutions of* (E) *tend to u_0 as $t \longrightarrow \infty$.*

On finite dimensional spaces one can use the Schauder theorem to establish the existence of solutions of (E_0) if $g(u) = o(\|u\|)$ as $\|u\| \longrightarrow \infty$ (see [1] for a related result) or if $g(u) = \epsilon h(u)$ where $\|h(u)\| \leqslant k(\|u\| + 1)$ and ϵ is small. If one has a priori bounds for solutions of (E) then one can eliminate the growth condition on g by making use of Miller's result and Theorem 1. One situation in which this idea can be applied is that of the following lemma which is given for $H = R^1$ in [3].

LEMMA. *Suppose that f in* (E) *is in $L_\infty((-\infty, \infty) : H)$. Suppose further that the conditions of Theorem* (1) *hold and that there exists an $\alpha > 0$ such that $e^{\alpha t}A(t)$ still satisfies the hypotheses of Theorem* (1). *Then there exists an $M > 0$ such that all solutions of* (E) *satisfy $\|u(t)\| \leqslant M$ for all $t \geqslant 0$.*

Similar results to those in this paper can be obtained for the differentiated version of (E). These are related to the results of [2].

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MAXIMA IN BROWNIAN EXCURSIONS

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Let $\{X(t), t \geqslant 0\}$ be the standard one-dimensional Brownian motion starting at 0. For $t > 0$ define

$$T(t) = \sup\{s \leqslant t | X(s) = 0\}; \qquad T'(t) = \inf\{s \geqslant t | X(s) = 0\};$$

$$L^-(t) = t - T(t); \qquad L(t) = T'(t) - T(t);$$

$$M^-(t) = \max_{T(t) \leqslant s \leqslant t} |X(s)|; \qquad M(t) = \max_{T(t) \leqslant s \leqslant T'(t)} |X(s)|.$$

The random time interval $(T(t), T'(t))$ is the *excursion interval straddling t*,

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