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THE LOOP SPACE PROBLEM AND ITS CONSEQUENCES

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0. **Introduction.** One of the key results in the study of the topology of Lie groups is the following theorem of Bott [2]:

THEOREM. *Let G be a simply connected Lie group. Then* $H_*(\Omega G; Z)$ *is torsion free.*

Bott subsequently coauthored a paper with Samelson [3] which uses this theorem to obtain extensive information about the homotopy and homology of Lie groups. Later, Araki [1] used this result to compute the mod p cohomology of the exceptional groups E_7 and E_8 over the Steenrod algebra. Bott's proof depends heavily on the existence of a differential structure on the Lie group.

Shortly after Bott proved this result, it was conjectured that the integral homology of the loops on a finite simply connected H-space should be torsion free. We resolve this conjecture for odd primes:

THEOREM 1. *Let X be a simply connected finite H-space. Then* $H_*(\Omega X; Z)$ *has no odd torsion.*

Actually, we prove this result in a much more general setting. Unlike Bott's proof, which relies heavily on the differential structure, our proof is purely homological and can be applied to H-spaces that do not even have the homotopy type of a finite complex.

I wish to thank Bill Browder, John Harper, Richard Kane, J. C. Moore and Alex Zabrodsky for many helpful discussions. I am especially indebted to Richard Kane for pointing out the theorems about the sparseness of the even generators in the mod p cohomology ring of an H-space.

1. **Statement of results.** For the remainder of the paper, X will be a two-connected H-space having the homotopy type of a CW complex with finitely many cells in each dimension. Furthermore, p will be an odd prime, and we will assume $QH^{\text{even}}(X; Z_p)$ is finite dimensional and $\beta_1 QH^{\text{even}}(X; Z_p) = 0$.

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Work of Browder [5] shows that any simply connected finite H-space satisfies the above conditions. It can be shown that if X is an H-space with a finitely generated mod p cohomology algebra and the Bockstein spectral sequence collapses after a finite number of steps, then X satisfies the conditions stated above. We have the following theorems:

THEOREM 2. *$H_*(\Omega X; Z)$ has no odd torsion.*

In the process of proving Theorem 2 we get

THEOREM 3. *$QH^{\text{even}}(X; Z_p) = \Sigma_{l=1}^{\infty} \beta_1 P^l QH^{2l+1}(X; Z_p)$ and $H^*(X; Z)$ has p-torsion of order at most p.*

We now generalize some theorems of Richard Kane [7].

DEFINITION. Let m have p-adic expansion

$$m = \sum_{s\geq 0}^{j} m_s p^s, \qquad m_j \neq 0,\ 0 \leq m_s < p.$$

Then:

m is "unary" if $m_s = 1$ for every $s \leq j$;
m is "binary" if $m_s = 0$ or 1 for every $s \leq j$;
m is "nonbinary" if m is not binary.

Let

$$v(k) = 1 + p + p^2 + \cdots + p^k, \qquad k \geq 1,$$

$$v(0) = 1, \qquad v(-1) = 0.$$

Note that if m is binary but not unary, m may be written

$$m = v(k) + p^l + \sum_{s>l}^{j} m_s p^s, \qquad m_{k+1} = 0,\quad l > k+1.$$

THEOREM 4. (a) *Let m be nonbinary. Then $QH^{2m}(X; Z_p) = 0$.*
(b) *Let m be binary but not unary;*

$$m = v(k) + p^l + \sum_{s>l}^{j} m_s p^s, \qquad l > k+1.$$

Then

$$QH^{2m}(X; Z_p) = P^{(m - v(k))/p} QH^{2v(k) + 2(m - v(k))/p}(X; Z_p).$$

Theorem 4 was proven by Kane when $H_*(X; Z_p)$ is associative and X is a simply connected finite H-space. We do not need these hypotheses. Theorem 4 may be used to show that the commutator of any two even primitives in $H_*(X; Z_p)$ is zero. Similarly, there are no homology primitive pth powers.

John Harper pointed out this last theorem:

THEOREM 5. *The kernel of the Hurewicz map,*

$$\Pi_n(X) \otimes Z_{(p)} \xrightarrow{h_n \otimes Z_{(p)}} H_n(X; Z) \otimes Z_{(p)},$$

is the p-torsion of $\Pi_n(X)$.

The proof of these theorems uses techniques developed by Zabrodsky [12] and myself [8], [10]. Details and proofs will appear elsewhere.

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