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CONVOLUTEURS OF H^p SPACES

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Nonperiodic analogues and generalizations of some results of Duren and Shields [1] are given. In the process, the key role played by the homogeneous Besov spaces and their images under the Fourier transform will be highlighted. Our results concern the following spaces in addition to the usual L^p space on R^n for $0 < p < \infty$.

Let ϕ be a smooth function (belonging to S, the space of rapidly decreasing functions) such that $\int \phi(x)\,dx = 1$. Set $\phi_t(x) = t^{-n}\phi(xt^{-1})$, and for f defined on R^n, call

$$u(x, t) = \phi_t * f(x), \qquad u^+(x) = \sup_{t>0} |u(x, t)|.$$

A function f defined on R^n belongs to $H^p(R^n)$, $0 < p < \infty$, if and only if $u^+ \in L^p(R^n)$ [2].

For $0 < \alpha < 1$, $f \in \Lambda^\alpha_{a,p}$ iff $(\int(\|\Delta_h f\|_a/|h|^\alpha)^p dh/|h|^n)^{1/p}$ is finite, where $\Delta_h f$ is the difference operator, $\Delta_h f(x) = f(x + h) - f(x)$. The spaces are defined for other values of α by the formula $R^\beta \Lambda^\alpha_{a,p} = \Lambda^{\alpha+\beta}_{a,p}$, where R^β is the Riesz potential of order β defined by closing the operator defined on a subset of S by $(R^\beta f)^\wedge(\xi) = |\xi|^{-\beta}\hat{f}(\xi)$. These spaces are homogeneous Besov spaces [6] in contrast to [9]; for $a = p = 2$, α a nonnegative integer, this is the space of tempered distributions for which all derivatives of order α are in L^2, with no other condition on the lower order derivatives or on the function itself. The characterization that we use most frequently is given in [6]; $f \in \Lambda^\alpha_{p,q}$ iff for k the smallest nonnegative integer greater than $\alpha/2$, if u denotes the temperature with initial value f, and if $M_p(u; t) = \|u(\cdot, t)\|_p$, then

$$\left(\int_0^\infty \left[t^{k-\alpha/2} M_p\left(\frac{\partial^k u}{\partial t^k}; t\right)\right]^q t^{-1}\,dt\right)^{1/q}$$

is finite. For example, $\Lambda^{n(1-1/p)}_{1,1}$ is the containing Banach space B^p of Duren and Shields [1].

Finally, we must consider the Fourier image of these spaces, the spaces

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$K^{\alpha}_{p,q}$, also defined in [6], and further characterized in [3], [7]. Let us denote by $v(\xi) = b_n|\xi|^n$ the volume of the ball of radius $|\xi|$. A function, defined and measurable on R^n, belongs to $K^0_{p,q}$ iff

$$\left[\sum_{-\infty}^{\infty}\left(\int_{2^k \leqslant |\xi| \leqslant 2^{k+1}} |f(\xi)|^p \, d\xi\right)^{q/p}\right]^{1/q}$$

is finite. The function f belongs to $K^{\alpha}_{p,q}$ iff $fv^{\alpha/n}$ belongs to $K^0_{p,q}$. Other characterizations, some of which are required in the proofs, are given in [3], [6], [7].

Recall that $Cv(X, Y) = \{k | \text{ for every } f \in X,\ k * f \in Y \text{ and } \|k * f\|_Y \leqslant B\|f\|_X\}$, while $M(X, Y) = \{\hat{k} | k \in Cv(X, Y)\}$. (In general, $FX = \{\hat{f} | f \in X\}$.) Then we have the following results.

THEOREM 1. *If* $0 < p < 1 \leqslant q \leqslant \infty$, $1 \leqslant s \leqslant q$, *then*

$$Cv(H^p, FL^q) = Cv(\Lambda^{n(1-1/p)}_{1,s}, FL^q) = \{k | \hat{k} \in K^{n(1/p-1)}_{q,\infty}\};$$

or equivalently, $M(H^p, FL^q) = K^{n(1/p-1)}_{q,\infty}$.

THEOREM 2. *If* $2 \leqslant q \leqslant \infty$, $1 \leqslant s \leqslant q$, *then*

$$Cv(H^1, FL^q) = Cv(\Lambda^0_{1,s}, FL^q) = \{k | \hat{k} \in K^0_{q,\infty}\}.$$

Theorem 2 generalizes the standard Hardy-Littlewood theorem [5] to R^n; the restriction $2 \leqslant q$ occurs because of considerations involving the Littlewood-Paley function and because of the inclusion $\Lambda^0_{1,1} \subseteq H^1 \subseteq \Lambda^0_{1,2}$, which is best possible. Thus Theorem 2 does not tell us about $Cv(H^1, FL^1)$; the best that can be said with these methods is

THEOREM 3. $K^0_{1,2} \subseteq M(H^1, FL^1) \subseteq K^0_{1,\infty}$.

These theorems are applied to give theorems on the growth rate of Fourier transforms of H^p functions, as well as Paley type theorems on the behaviour of Fourier transforms on lacunary sets.

The techniques also give results on the convoluteurs of H^p spaces.

THEOREM 4. *If* $0 < p < 1 \leqslant q < \infty$, *then*

$$Cv(H^p, H^q) = \Lambda^{n(1/p-1)}_{q,\infty}.$$

In particular, using the results of [5],

$$M(H^p, H^2) = K^{n(1/p-1)}_{2,\infty}.$$

The next results, which include most of the previously known sufficient conditions for convoluteurs in terms of Besov spaces, show that the appearance of Besov spaces as necessary or sufficient conditions for convoluteurs can be traced back to Sobolev's theorem.

THEOREM 5. *If* $1 \leqslant p, q, r \leqslant \infty$, *and* α *is real,*

$$Cv(L^p, L^q) \subseteq Cv(\Lambda^\alpha_{p,r}, \Lambda^\alpha_{q,r}).$$

This result sheds light on the theorems and counterexamples of [10], (see also [6], [7]). Theorem 1 of [10] follows easily now from Sobolev's inequality (and thus is essentially equivalent to it). The counterexamples are to be expected because one tries to force an element of $Cv(L^p, L^q)$ to map Besov spaces of (∞, ∞) type rather than of the (p, q) type. Theorem 5 may also be used to provide a simpler proof of the basic inclusion result for homogeneous Besov spaces and to show that Calderón-Zygmund operators preserve the Besov spaces.

THEOREM 6. *If* $1 < p \leqslant 2 \leqslant q < \infty$, *then*

$$Cv(L^p, L^q) = Cv(\Lambda^0_{p,2}, \Lambda^0_{q,2}).$$

This result generalizes the result of the Appendix of [10].

The technique used in all cases is to apply the assumed convoluteur to ϕ a member of the family of test functions of derivatives of the fundamental solution of the heat equation, and use the various equivalent norms on the spaces. The restriction $1 \leqslant q$ imposed throughout can probably be dropped by using the Besov spaces $\Lambda^\alpha_{p,r}$ for $p < 1$ [8], along with the corresponding $K^\alpha_{p,q}$ spaces [3].

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A COHOMOLOGICAL STRUCTURAL THEOREM FOR TOPOLOGICAL ACTIONS OF Z_2-TORI ON SPACES OF Z_2-COHOMOLOGY TYPE OF SUCCESSIVE FIBRATION OF PROJECTIVE SPACES

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Let X be a given G-space and $X \to X_G \to B_G$ be the universal bundle with X as its typical fibre. We shall consider the *ordinary cohomology of the total space* $H^*(X_G)$ as the *equivariant cohomology of* X, namely, we shall take $H^*_G(X) = H^*(X_G)$ as the definition of the equivariant cohomology theory. In case G are *elementary abelian groups* (i.e., tori or Z_p-tori), several fundamental cohomological splitting theorems are formulated and proved in [1], [2] which establish definitive, neat correlations between the *cohomological orbit structures* (e.g., $H^*(F)$, orbit types, etc.) of the given G-space X and the various *ideal theoretical invariants of* $H^*_G(X)$. In the simplest cases that $H^*(X)$ are generated by a single generator (e.g., spheres, projective spaces), the ideals occur in such cohomological splitting theorems are automatically principal ideals. Therefore the cohomological structural theorems for topological ac-

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