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LOOP SPACES AND FINITE ORTHOGONAL GROUPS[1]

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In this note we announce the results of our computations of the mod 2 homology of the orthogonal groups $O(n, \mathbf{F}_q)$ over finite fields $\mathbf{F}_q$ of characteristic $p \neq 2$. We have obtained a 2-local equivalence between the infinite loop space associated with these orthogonal groups and the homotopy fiber $JO(q)$ of the map $(\psi^q - 1)$: $BO(\mathbf{R}) \longrightarrow BSO(\mathbf{R})$, where ψ^q is the Adams operation. For $q \equiv \pm 3 \pmod 8$, these spaces $JO(q)$ are of considerable geometric interest, since $\pi_* JO(q)$ is essentially the image of J_*: $\pi_* SO(\mathbf{R}) \longrightarrow \pi_* SF$ at the prime 2. Here J: $SO(\mathbf{R}) \longrightarrow SF$ is the J-homomorphism of G. Whitehead.

Since the Whitney sum induces an infinite loop space structure on $JO(q)$, we can define Dyer-Lashof operations on its homology. We have computed $H_*(JO(q), \mathbf{Z}_2)$ as an algebra over the Dyer-Lashof algebra.

Our main results are as follows:

THEOREM 1. *There is an equivalence of infinite loop spaces*

$$\Gamma_0 B\bar{O}(\mathbf{F}_q)_{(2)} \xrightarrow{\simeq} JO(q)_{(2)}.$$

Here $\Gamma_0 B\bar{O}(\mathbf{F}_q)$ denotes the 0-component of the group completion of $\amalg_{n=0}^{\infty} BO(n, \mathbf{F}_q)$. See May [3] for details.

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We should point out that in a recent paper [1], E. M. Friedlander, using the methods of etale homotopy theory, has established that there is an equivalence of spaces $\Gamma_0 B\bar{O}(\mathbf{F}_q) \xrightarrow{\simeq} JO(q)$. However there seems to be no apparent way of showing that his map is an infinite loop map.

Using the equivalence of Theorem 1, we can now compute homology operations in $H_*(JO(q), \mathbf{Z}_2)$ by the methods employed in Priddy [4]. We obtain the following result:

THEOREM 2. *$H_*(JO(q), \mathbf{Z}_2) = \mathbf{Z}_2[\bar{v}_1, \bar{v}_2, \ldots] \otimes E\{\bar{u}_i \mid i \geqslant 1\}$, where the $\bar{v}_i$'s are explicitly chosen classes which map onto the standard generators $\{\bar{e}_i\}$ of $H_*(BO(\mathbf{R}), \mathbf{Z}_2)$ and the $\bar{u}_i$'s come from the standard generators u_i of $H_*(SO(\mathbf{R}), \mathbf{Z}_2)$. The action of the Dyer-Lashof algebra on $H_*(JO(q), \mathbf{Z}_2)$ is given by*

(1) $$Q^n(\bar{u}_k) = \sum_{a+b+c=n+k} (k-a, n-k-b-1)\bar{u}_a * \bar{u}_b * \bar{u}_c;$$

(2) $$Q^n(\bar{v}_k * [1]) = \sum_{a+b+c+d=n+k} (n-k-1, a+c-n)\bar{v}_a * \bar{v}_b * \bar{v}_c * \bar{v}_d * [2]$$

if $q \equiv \pm 3 \pmod 8$;

(3) $$Q^n(\bar{v}_k * [1]) = \sum_{a+b=n+k} (n-k-1, a-n)\bar{v}_a * \bar{v}_b * [2]$$

if $q \equiv \pm 1 \pmod 8$.

The generators $\{\bar{v}_i\}$ of Theorem 2 can be chosen in two different ways corresponding to two different lifts into $JO(q)$ of the inclusion $\mathbf{R}P^\infty = BO(1, \mathbf{R}) \longrightarrow BO(\mathbf{R})$. There is an involution Φ: $JO(q)_{(2)} \longrightarrow JO(q)_{(2)}$ of infinite loop spaces which sends one lift to the other. In $\Gamma_0 B\bar{O}(\mathbf{F}_q)$ this involution is induced by conjugation by the matrix $\left(\begin{smallmatrix} -a & b \\ b & a \end{smallmatrix}\right)$ where a, b are elements of $\mathbf{F}_q$ such that $a^2 + b^2$ is not a square.

We remark here that our computations produce a much simpler derivation of the formula for the Dyer-Lashof operations on $H_*(SO(\mathbf{R}), \mathbf{Z}_2)$ than the one in Kochman [2].

To obtain generators for the homology groups of $O(n, \mathbf{F}_q)$, we proceed as follows. Since $O(1, \mathbf{F}_q) = \mathbf{Z}_2$, $H_k(BO(1, \mathbf{F}_q), \mathbf{Z}_2) = \mathbf{Z}_2$, and we denote by v_k the generator of this group. Let $T \subset O(2, \mathbf{F}_q)$ be generated by $\{B, -B\}$ where $B = \Phi\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $T = \mathbf{Z}_2 \times \mathbf{Z}_2$, and we denote by $v_{k,l}$ the image of $x_k \otimes x_l$ under

$$H_k(\mathbf{R}P^\infty, \mathbf{Z}_2) \otimes H_l(\mathbf{R}P^\infty, \mathbf{Z}_2) \longrightarrow H_{k+l}(BT, \mathbf{Z}_2)$$
$$\longrightarrow H_{k+l}(BO(2, \mathbf{F}_1), \mathbf{Z}_2).$$

Then we have

THEOREM 3. *As an algebra,* $\bigoplus_{n=0}^{\infty} H_*(BO(n, \mathbf{F}_q), \mathbf{Z}_2)$ *is commutative with generators* v_i, $v_{i,j}$, $i, j \geqslant 0$, *subject to the relations*

(1) $v_{i,j} = v_{j,i}$,

(2) $v_{i,i} = v_i^2$,

(3) $v_{i,j}v_{k,l} = v_{i,k}v_{j,l}$.

By including the extraordinary orthogonal groups $O^-(n, \mathbf{F}_q)$ (= automorphisms of the quadratic space $(\mathbf{F}_q^n, Q^-)$ where $Q^-(x_1, \ldots, x_n) = \mu x_1^2 + x_2^2 + \cdots + x_n^2$ and $\mu \in \mathbf{F}_q$ is a nonsquare) the relations of Theorem 3 simplify considerably. Since $O^-(1, \mathbf{F}_q) = \mathbf{Z}_2$, let $v_k^- \in H_k(BO(1, \mathbf{F}_q), \mathbf{Z}_2) = \mathbf{Z}_2$ denote the generator. Let $O^+(n, \mathbf{F}_q) = O(n, \mathbf{F}_q)$ (= automorphisms of the quadratic space $(\mathbf{F}_q^n, Q^+)$ where $Q^+(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$) and set $v_k^+ = v_k$. There are isomorphisms $(\mathbf{F}_q^n, Q^\pm) \oplus (\mathbf{F}_q^m, Q^-) \approx (\mathbf{F}_q^{n+m}, Q^\mp)$.

THEOREM 4. *As an algebra*

$$H_*(BO(O, \mathbf{F}_q), \mathbf{Z}_2) \oplus \bigoplus_{n=0}^{\infty} H_*(BO^\pm(n, \mathbf{F}_q), \mathbf{Z}_2)$$

is commutative with generators v_i^+, v_i^-, $i \geqslant 0$ *subject to the single relation* $(v_i^+)^2 = (v_i^-)^2$.

The generator $v_{i,j}$ of Theorem 3 is seen to decompose as $v_{i,j} = v_i^- v_j^-$.

Proofs and additional details will appear in our forthcoming paper.

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