

*Kähleriennes*, Hermann & Cie, Paris, 1958. Read together, these books should be enough to explain the Hermitian algebra. This algebra is applied together with the Hodge theory to prove some of the pretty classical results on Kähler manifolds. For example the Hodge decomposition theorem, decomposing a cohomology class into a sum of harmonic  $(p, q)$  forms is proved. The Lefschetz decomposition theorem is also proved, and the Hodge-Riemann bilinear relations are discussed. This is done on the primitive cohomology of a Kähler manifold and an example is produced to show that the result is only valid on the primitive cohomology. Unfortunately there is a mistake in the computation on p. 200 which invalidates the example. There are a fair number of misprints in this chapter, but they generally do not detract from its quality. The reviewer found that after one gets past the algebra the rest is well written and gives an interesting introduction to the papers of Griffiths on periods of integrals on algebraic manifolds.

The discussion of Chapter VI is directed toward a proof of Kodaira's theorem that a Hodge manifold is projective. The proof follows Kodaira's original proof. One first proves Kodaira's vanishing theorem, and then makes an application of this result to the blow up of the Hodge manifold to produce enough sections to give an embedding in complex projective space. The proof of the vanishing theorem differs from Kodaira's in that Nakano's inequality is the crucial ingredient. The reviewer thoroughly enjoyed this chapter and found the exposition to be very clear. There are some confusing misprints in the discussion of the canonical bundle but it is an easy task to correct them. The reader should compare this chapter with the last few pages of the book by Gunning and Rossi, *Analytic functions of several variables*, Prentice-Hall, Englewood Cliffs, N.J., 1965, where a discussion of Grauert's proof of this theorem is given.

The topics treated in the book under review are fundamental. Every complex analyst should know (or learn) this basic material, and Wells' book is a good reference for these essential results about complex manifolds.

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*Topics in analytic number theory*, by Hans Rademacher, Die Grundlehren der math. Wissenschaften, Band 169, Springer-Verlag, Berlin, 1973, ix+320 pp.

*Topics in analytic number theory* by Hans Rademacher covers all the classical aspects of a subject which is presently undergoing a revolution. According to the editors, Professor Rademacher had been working on this

book for 25 years. When he died in 1969 there was a complete manuscript available which has now fortunately appeared in book form. This is a beautiful book whose value is greatly enhanced by the fact that there is no other modern book like it. Although it contains a proof of the prime number theorem, the main thrust of the book is number theory related to elliptic functions and modular functions.

The book begins with Bernoulli polynomials and the Euler-MacLaurin sum formula. It then moves through Mellin transforms and the Poisson summation formula. As an example of these results, let the Riemann zeta function be given by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

for  $\text{Re } s > 1$  and let the theta function  $\vartheta_3(v|\tau)$  of Jacobi be given by

$$\vartheta_3(v|\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n v)$$

which is defined for all complex  $v$  and all  $\tau$  with  $\text{Im } \tau > 0$ . The connection between these functions is the Mellin transform formula,

$$(1) \quad \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} \int_0^{\infty} [\vartheta_3(0|iu) - 1] u^{(s/2)-1} du,$$

as may be seen by integrating term by term. The extension of  $\zeta(s)$  to the entire plane and the functional equation for  $\zeta(s)$ ,

$$(2) \quad \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

follows from the Jacobi transformation equation,

$$(3) \quad \vartheta_3(0|-1/\tau) = (-i\tau)^{1/2} \vartheta_3(0|\tau)$$

(the square root being the principal value). However, conversely (2) implies (3) by the theory of Mellin transforms.

After a short, but interesting chapter on the prime number theorem<sup>1</sup> and its connection with zeros of  $\zeta(s)$ , the book turns to a detailed study of Eisenstein series. Let  $\omega_1$  and  $\omega_2$  be complex numbers whose quotient,  $\tau = \omega_2/\omega_1$ , has positive imaginary part. Let  $\Lambda$  be the lattice generated by  $\omega_1$  and  $\omega_2$  so that  $\Lambda$  consists of all  $\omega = m_1\omega_1 + m_2\omega_2$  where  $m_1$  and  $m_2$  are integers. Eisenstein series are closely related to the Weierstrass  $\wp$ -function,

<sup>1</sup> There is an error in Rademacher's proof of the prime number theorem. The absolute value sign on the right side of the equation immediately following (49.6) on p. 100 is not justified; the correct expression is minus what is inside the absolute value sign. This necessitates a change in the choice of  $\sigma - 1$  two lines later from  $\log^{-9} |t|$  to  $(2AB)^{-4} \log^{-9} |t|$ . With this change, (49.7) and (49.8) follow.

$$f(u) = \frac{1}{u^2} + \sum'_{\omega \in \Lambda} \left[ \frac{1}{(u - \omega)^2} - \frac{1}{\omega^2} \right]$$

where  $\sum'$  means  $\omega=0$  is not used. This is a doubly periodic function of  $u$  whose periods are the elements of  $\Lambda$ . Its Laurent series about  $u=0$  is

$$f(u) = \frac{1}{u^2} + \sum_{k=2}^{\infty} (2k-1)G_{2k}u^{2k-2}$$

where  $G_r$  is the Eisenstein series

$$(4) \quad G_r = G_r(\omega_1, \omega_2) = \sum'_{\omega \in \Lambda} \omega^{-r} \quad (r \geq 3).$$

This series only converges absolutely for  $r > 2$  and  $G_r = 0$  if  $r$  is odd since if  $\omega$  is in  $\Lambda$ , so is  $-\omega$ .

Suppose

$$(5) \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathcal{Z}),$$

so that  $A$  is a matrix of integers of determinant 1. Rademacher calls  $\text{SL}(2, \mathcal{Z})$  the modular group. If  $\omega'_1, \omega'_2$  and  $\tau'$  are given by

$$(6) \quad \begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix} = A \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}, \quad \tau' = \frac{\omega'_2}{\omega'_1} = \frac{a\tau + b}{c\tau + d}$$

then  $\omega'_1$  and  $\omega'_2$  generate  $\Lambda$  also (and  $\text{Im } \tau' > 0$ ) and so

$$(7) \quad G_{2k}(\omega'_1, \omega'_2) = G_{2k}(\omega_1, \omega_2).$$

Further for any  $\lambda \neq 0$ ,

$$(8) \quad G_{2k}(\lambda\omega_1, \lambda\omega_2) = \lambda^{-2k}G_{2k}(\omega_1, \omega_2).$$

The properties (7) and (8) make  $G_{2k}$  a (homogeneous) modular form of dimension  $-2k$ .

Set

$$G_{2k}(\tau) = G_{2k}(1, \tau) = \omega_1^{2k}G_{2k}(\omega_1, \omega_2).$$

From (7) and (8) we see that for  $A$  given by (5),

$$(9) \quad G_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k}G_{2k}(\tau).$$

The property (9) makes  $G_{2k}(\tau)$  an (inhomogeneous) modular form of dimension  $-2k$ . Since  $G_{2k}(\tau+1) = G_{2k}(\tau)$ , we have a Fourier series expansion which, with

$$(10) \quad x = e^{2\pi i\tau},$$

turns out to be

$$(11) \quad G_{2k}(\tau) = \frac{(-1)^k \cdot 2(2\pi)^{2k}}{(2k-1)!} \left[ -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n)x^n \right]$$

where  $B_{2k}$  is the  $(2k)$ th Bernoulli number and  $\sigma_r(n) = \sum_{d|n} d^r$ .

Because the series in (4) does not converge absolutely for  $r=2$ ,  $G_2$  has not yet been defined. The book follows Hecke's method of defining  $G_2$  by letting  $G_2(\omega_1, \omega_2, s) = \sum'_{\omega \in \Lambda} \omega^{-2} |\omega|^{-s}$  which converges absolutely for  $s > 0$  and then defining

$$G_2(\omega_1, \omega_2) = \lim_{s \rightarrow 0^+} G_2(\omega_1, \omega_2, s).$$

The properties (7) and (8) are clear for  $G_2$  as well and we get the expansion (11) with  $k=1$  except with one extra term:

$$(12) \quad G_2(\tau) = -2 \cdot (2\pi)^2 \left[ -\frac{B_2}{4} + \sum_{n=1}^{\infty} \sigma_1(n)x^n \right] - \frac{2\pi i}{\tau - \bar{\tau}}.$$

Thus  $G_2(\tau)$  is not an analytic function of  $\tau$ . Nevertheless, the transformation formula (9) is still valid for  $k=1$  and if we integrate it with respect to  $\tau$  we get the logarithm of the Dedekind  $\eta$ -function,

$$(13) \quad \eta(\tau) = x^{1/24} \prod_{n=1}^{\infty} (1 - x^n),$$

as well as its transformation formula: if  $A$  is given by (5) with  $c > 0$ ,

$$(14) \quad \log \eta \left( \frac{a\tau + b}{c\tau + d} \right) = K + \frac{1}{2} \log[-i(c\tau + d)] + \log \eta(\tau),$$

where  $K=K(A)$  is a constant of integration. In particular, if  $c > 0$ ,

$$(15) \quad \eta \left( \frac{a\tau + b}{c\tau + d} \right) = e^K [-i(c\tau + d)]^{1/2} \eta(\tau).$$

A special case of great importance is,

$$(16) \quad \eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau);$$

here  $K=0$  as may be seen by setting  $\tau=i$ . This case together with the fact that  $\eta(\tau+1) = e^{2\pi i/24} \eta(\tau)$  (as may be seen from (13)) shows that  $e^K$  in (15) is always a 24th root of unity.

The book turns next to evaluating  $K$  in (14). For this purpose, the theory of Dedekind sums is discussed. This approach is so standard today that it has been practically forgotten that the problem of writing  $e^K$  in closed form was solved long ago. Happily, Rademacher includes these results also.

Although derived here from properties of Dedekind sums, it should be noted that once the correct value of  $e^K$  is suspected, it may be easily proved since it behaves as it should when  $\tau$  is replaced by  $\tau+1$  and  $-1/\tau$ .

From here, the book turns to a detailed study of theta functions. Transformation formulae, product formulae and connections with the  $\eta$ -function are covered. As an example, we see from (3) that  $\vartheta_3(0|\tau)$  behaves in a manner similar to  $\eta(\tau)$  and so it should be no surprise that a relation such as

$$\vartheta_3(0|\tau) = e^{-(\pi i/12)} \eta[(\tau+1)/2]^2 / \eta(\tau)$$

holds; however, thanks to (13), this reflects a remarkable product expansion for  $\vartheta_3(0|\tau)$ . Jacobi's method of producing elliptic functions from theta quotients is briefly mentioned in Chapter 11 but not developed to any degree. Although usually not known by the casual student of elliptic functions, Jacobi's method is particularly well suited to the construction of multiply periodic meromorphic functions of  $n$  complex variables. The interested reader is referred to Siegel [16, Chapter 5]. In Rademacher's book, the relation with elliptic functions is used to derive various identities, such as formulae for the number of representations of an integer  $n$  by a sum of  $k$  squares with  $k$  even,  $2 \leq k \leq 12$ .

The book next turns to a study of formal power series. A large number of relations involving partitions of various types may be read off from identities between certain series. Although these identities may be considered as being between analytic functions, they may often be derived purely formally without regard to convergence. In these sections, the book covers such things as the Jacobi triple product theorem and the Rogers-Ramanujan identities.

With formal power series out of the way, the book turns to the circle method. Let  $p(n)$  be the number of partitions of  $n$  as a sum  $n = n_1 + n_2 + \dots + n_k$  with  $1 \leq n_1 \leq \dots \leq n_k$  and  $k$  arbitrary. Then it is easily seen purely formally that

$$f(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = 1 / \prod_{n=1}^{\infty} (1 - x^n)$$

and in fact, this identity is valid for all complex  $x$  with  $|x| < 1$ . This brings up a natural connection with the  $\eta$ -function and we have

$$f(x) = x^{1/24} / \eta(\tau)$$

where  $x$  is given by (10) as before.

The transformation formula (15) (with  $c=k$ ,  $d=-h$ ) shows that  $f(x) \rightarrow \infty$  for  $\tau = h/k + iy$  and  $y \rightarrow 0^+$  so that  $|x|=1$  is a natural boundary for  $f(x)$ . But the same transformation formula shows exactly how  $f(x)$  behaves as  $y \rightarrow 0^+$  and by analyzing this behavior for all  $h/k$ , Hardy and Ramanujan

found an asymptotic series for  $p(n)$  which, although later found to be divergent by Lehmer, actually comes within  $\frac{1}{2}$  of  $p(n)$  for large  $n$  when cut off at on the order of  $n^{1/2}$  terms. This was explained by Rademacher who altered the Hardy-Ramanujan approach sufficiently to obtain a convergent series for  $p(n)$  which essentially agrees with the Hardy-Ramanujan series for the first  $n^{1/2}$  terms. The book presents Rademacher's convergent series for  $p(n)$ .

What can be done for  $1/\eta(\tau)$  can be done for other modular forms of positive dimension. The book restricts itself to the case of entire modular forms (modular forms which are regular in the upper half plane) with a finite number of terms in the principal part of the expansion at  $\infty$ . One gets an exact formula for all the coefficients of the form in terms of the principal part. In particular, the principal part determines the form uniquely but there are easier ways of proving this. Here the book ends.

The editors are to be congratulated on an excellent job of proofreading. Considering the complexity of the subject and notation, the equations as written are remarkably trustworthy. [As in all books, there are exceptions; for example, the table of Bernoulli numbers on p. 10 contains a counter-example to the von Staudt-Clausen theorem three lines later; fortunately this is because  $B_{14} = 7/6$  and not  $6/7$  as printed. Also on p. 148, case (ii) of the reciprocity formula for Dedekind sums as printed is a repeat of case (i).] Nevertheless, the casual quoter should be aware of certain differences in terminology and notation between this book and other sources. Mathematicians have a nasty habit of slightly altering the meaning of a word or symbol. For a few years, they and their followers say that they are using the new meaning, but gradually it becomes accepted and the explanation is no longer offered. From that point on, the confusion is total.

Professor Rademacher wrote his book with the classical words and notation. When we see in the book that a modular form should be analytic for  $\text{Im } \tau > 0$ , we nod our heads in agreement not knowing that in this book such a function can have poles. When we see a theta function expanded in a power series in  $q$  we are happy, not noticing that for Rademacher  $q = e^{\pi i \tau}$  and not  $e^{2\pi i \tau}$  (which is classically denoted by  $x$ ) as is often the case today. A modular form in this book has no condition at  $\infty$  as it would for modern writers. A modular form of positive dimension now has positive degree. It also has negative weight and, of course, some people's weights are twice those of others. And so it goes. It is probably too late to undo the carnage wrought in this subject but perhaps it will serve as a warning each time we are tempted to slightly change some established terminology for our own temporary convenience.

Many of the topics treated in the book may be treated in other ways. The Dedekind  $\eta$ -function is a case in point. For instance, there is

Kronecker's limit formula (see [17, Chapter 1] for several interesting versions),

$$(17) \quad \lim_{s \rightarrow 1^+} \left\{ \sum'_{\omega \in \Lambda} \left[ -i \det \begin{pmatrix} \omega_2 & \bar{\omega}_2 \\ \omega_1 & \bar{\omega}_1 \end{pmatrix} / |\omega|^2 \right]^s - \frac{2\pi}{s-1} \right\} \\ = 4\pi\gamma - 4\pi \log\{[-i(\tau - \bar{\tau})]^{1/2} |\eta(\tau)|^2\},$$

where  $\gamma$  is Euler's constant. The left-hand side of (17) is clearly invariant under the modular group and homogeneous in  $\omega_1$  and  $\omega_2$  with degree 0. Thus with  $\tau'$  given by (6),

$$\log\{[-i(\tau' - \bar{\tau}')]^{1/2} |\eta(\tau')|^2\} = \log\{[-i(\tau - \bar{\tau})]^{1/2} |\eta(\tau)|^2\}.$$

Since  $\text{Im } \tau' = (\text{Im } \tau) / |c\tau + d|^2$ , this shows that the real parts of both sides of (14) agree (with  $\text{Re } K = 0$ ) and (14) follows.

The proof of (17) is by the same methods as the book's proof of (14) and of course the left side of (17) resembles the manner in which  $G_2$  was defined. But the resemblance is more than just superficial. With  $\omega_1 = 1$ ,  $\omega_2 = \tau$ , an individual term in (17) corresponding to  $\omega = m + n\tau$  at  $s = 1$  is

$$\frac{-i(\tau - \bar{\tau})}{(m + n\tau)(m + n\bar{\tau})} = \frac{i}{n} \left( \frac{1}{m + n\tau} - \frac{1}{m + n\bar{\tau}} \right).$$

The derivative of this with respect to  $\text{Im } \tau$  is just  $1/(m + n\tau)^2 + 1/(m + n\bar{\tau})^2$ . Thus from the point of view of Kronecker's limit formula, it is no surprise that (14) can be proved by expanding  $G_2$  in a series in  $x = e^{2\pi i\tau}$  and integrating.

There are other proofs of (16) besides the one in the book and the Kronecker limit formula proof. Among the more interesting are Siegel's proof [18] of (16) and Rademacher's generalization [9] to (14) with  $K$  being given in terms of Dedekind sums. There is also Chowla's proof [3, pp. VI-9 to VI-13] which was later rediscovered and popularized by Weil [19]. Dedekind sums are closely connected with the  $\eta$ -function and there are a multitude of proofs of the reciprocity formula and generalizations. Among the more useful generalizations is the one of Meyer [7] (see also Rademacher [10]) which has applications to generalizations of Kronecker's limit formula. There is an excellent monograph on Dedekind sums by Rademacher and Grosswald [14] which contains many further references.

The Rogers-Ramanujan identities and generalizations have been a continuing source of mathematical papers. Recent papers include the survey article by Alder [1], and the memoir by Andrews [2]. On the other hand, not much has been done recently with the circle method and coefficients of modular forms. However, there were several results around

1940 that were not referenced in the book (not even Rademacher's original 1937 paper [11] on  $p(n)$  is referenced). Most of the results of the last sections of the book can be found in the paper by Rademacher and Zuckerman [15], which in fact goes somewhat further than the book. These results were in turn generalized in two directions by Zuckerman; in one direction, one can consider modular forms on subgroups of the modular group [20] and in another direction, one can allow modular forms which have poles inside the upper half plane [21].

There are two further questions that arise from the Rademacher-Zuckerman paper. Given the proposed principal part of an entire modular form with a given multiplier system, they find a formula for all the remaining coefficients in its expansion. This determines the form uniquely if it exists at all. The question of whether or not there exists such a form was considered by Knopp [5]. There is also the problem of modular functions which is the limiting case of forms of dimension 0. Rademacher and Zuckerman note that when their method is applied to the  $j$ -function, they get a convergent series analogous to that for  $p(n)$  but this time it is not clear that these series converge to the coefficients of  $j(\tau)$ . Rademacher [12] proved in 1938 that indeed the circle method does give a convergent infinite series for the coefficients of the  $j$ -function (this same series had already been discovered in 1932 by Petersson [8] by a completely different method).

This completes a summary of some of the main subjects in *Topics in analytic number theory* and some further related references. However, we have not explained why the theory of modular forms is presently so popular. Algebraic geometers have been interested because of the Weil conjecture relating certain modular forms of weight 2 (dimension  $-2$ ) on  $\Gamma_0(N)$  (the subgroup of  $SL(2, \mathbf{Z})$  given by all  $A$  in (5) with  $c \equiv 0 \pmod{N}$ ) with elliptic curves. The same conjectural relations between certain Artin  $L$ -series and modular forms of weight 1 on  $\Gamma_0(N)$  have been formulated more recently by Langlands. In both of these instances, the relation is provided via the Mellin transform and is a direct analogue of the relation between  $\zeta(s)$  and  $\vartheta_3(0|\tau)$  in (1) above. Critical to this theory is the study of Hecke operators which provide a connection between Dirichlet series with Euler products and modular forms which are eigenfunctions of the Hecke operators. There is a marvelous survey article on analytic number theory by Rademacher [13] in 1942 which should be read by everyone. Among other topics, it provides a brief introduction to Hecke operators.

For the modern developments, the reader is referred to the proceedings of the 1972 summer school on modular functions held in Antwerp. Three volumes have appeared [6] and a fourth volume of tables should appear



someday. Since the Antwerp summer school, Deligne and Serre [4] have proved a converse to Langland's result. Any modular form of weight one on  $\Gamma_0(N)$  of a certain type must be related via the Mellin transform to an entire Artin  $L$ -series. Using this result, Tate and his students (unpublished) have just found a new modular form of weight one on  $\Gamma_0(133)$  (the first form of weight one not known to Hecke) and with it an entire Artin  $L$ -series which was not known to be entire by the usual group theoretic methods. The next few years promise to be very exciting ones in this field.

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*Uniform distribution of sequences*, by L. Kuipers and H. Niederreiter, Wiley, New York, 1974, xiv+390 pp., \$24.50

The theory of uniform distribution started with Hermann Weyl's celebrated paper of 1916 titled *Ueber die Gleichverteilung von Zahlen mod Eins* [13]. In its initial stage the theory was deeply rooted in diophantine approximations. Later the subject became a meeting ground for number theory, probability theory, functional analysis and topological algebra. The vast literature on uniform distribution is therefore widely spread. The existing surveys, for example [6], [1], [7], give only a partial introduction to the theory. It is a very valuable enrichment of the mathematical literature that a book has been published which is at the same time an easily accessible introduction to the subject and an almost complete account of it.

Writing a book for both beginners and researchers in a field is an almost impossible task. The authors show that it was not impossible in this case. Firstly, the basic concepts and ideas of the theory are mostly elementary. Secondly, by proving only the main results, by inserting references and additional results in rather lengthy notes at the end of each section and by adding many exercises of various sorts to each section, the authors succeed in describing the underlying principles of the theory in such a way that readers neither get lost in generalizations nor are drowned in technicalities. Thirdly, the finish of the book is excellent; the text is well got-up and at the end there is an extensive bibliography and further a list of symbols and abbreviations, an author index and a subject index.

An indication of the contents might explain the subject and the scope of the book. Chapter 1 deals with the qualitative aspects of uniform distribution modulo one (u.d. mod 1). A sequence  $(x_n)_{n=1}^{\infty}$  of real numbers is u.d. mod 1 if and only if for every continuous function  $f: [0, 1] \rightarrow \mathbf{R}$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\{x_n\}) = \int_0^1 f(x) dx,$$

where  $\{x\} = x - [x]$  denotes the fractional part of  $x$ . This implies that  $(x_n)_{n=1}^{\infty}$  is u.d. mod 1 if and only if the proportion of the numbers  $x_n$  in