Topics in complex function theory, by Carl L. Siegel, Interscience Tracts in Pure and Applied Mathematics, No. 25, Vol. I, 1969, ix+186 pp., (Translator: A. Shenitzer and D. Solitar); Vol. II, 1971, 256 pp., (Translator: A. Shenitzer and M. Tretkoff); Vol. III, 1973, ix+244 pp., (Translator: A. Shenitzer and M. Tretkoff).

The late Solomon Lefschetz once defined a topological space as "a space where every point has a neighborhood and every neighborhood has a point." On page 30 of volume II of the set under review here we find: "Let G be a space of points p for which neighborhoods are defined in the usual way." The first quotation is brought forth because it makes explicit the underlying feeling of the second. It is that in these three books, the modern spirit of abstraction is given a definitely secondary role, consciously and deliberately pushed out of the way for the concrete development of the area which these books treat, that of algebraic functions over the complex numbers and of automorphic functions and forms. In the reviewer's opinion these books are an excellent treatment of that subject, well-seasoned with numerous examples and important special results. In spite of the fact that there may be slicker treatments of various parts of the subject on the market, the relatively uninitiated graduate student who wants an introduction, as well as the specialist who wants to look up complete proofs for "well-known" facts whose proof is difficult to find elsewhere, would be well advised to turn to this set. This is not to say it has no faults, but the faults are relatively minor, and those which in the opinion of the reviewer might deserve the reader's attention, apart from language or proofreading (which is lax in places), are those which stem from Siegel's apparent preoccupation with *not* being slick (in the bad modern sense as he clearly views it). But above all, these volumes are packed full of important standard results in the field, and this is fundamental in evaluating them.

And now for an analysis of the contents by the numbers.

Chapter 1, vol. I, is on elliptic functions and commences with a section on the very old result about doubling the arc of a lemniscate. This is a good place to start because it is elementary and gives a good notion of elliptic integrals and of a problem that is connected with the beginnings of complex multiplication. This is followed by a natural sequel on the Euler addition theorem. Then, in preparation for a broader treatment of elliptic functions and elliptic integrals, there follow sections on analytic continuation and Riemann regions or, as the reviewer may sometimes refer to them, Riemann surfaces. Particular attention is given to the Riemann surface of the square root of a quartic polynomial, which leads naturally into the subject of elliptic integrals of the first kind and to the inverse function, i.e., a certain doubly periodic, or elliptic, function. It is then shown that the universal covering surface of that Riemann surface is nothing else than the complex plane on which the fundamental group of the original Riemann surface acts by the translations of a period lattice; that lattice is generated by the basic periods of an elliptic integral of the first kind, i.e., by the integrals of $dx/\sqrt{(Q(x))}$ around a suitable pair of paths generating the fundamental (or first homology) group, where $Q(x)$ is the quartic polynomial. The bulk of the rest of the chapter is a good standard analytical treatment of the elliptic functions themselves, the classical facts about their poles, zeros, and residues, their partial fraction expansions according to Mittag-Leffler, their Laurent expansions about poles, and the differential equations they satisfy, with special attention to the normalized Weierstrass $\wp$-function. After a section on the explicit algebraic formulation of the addition theorem, Siegel concludes this chapter with a section on degenerate elliptic functions, which arise on letting one of the fundamental periods tend to infinity. If one were to take this chapter as material for a second semester course on functions of a complex variable or Riemann surfaces, he might want to "slick up" the treatment of certain topological matters, for example, those dealing with the fundamental group. The same remark broadly applies to a few other places in the three volumes, where, granted Siegel's basic philosophy, there might be room for compromise with modernity.

Chapter 2 has as its central theme the proof of the so-called uniformization theorem, which states that a certain function with a simple pole on a simply-connected Riemann surface, constructed by means of Dirichlet's principle, effects a conformal mapping of the Riemann surface onto a slit region equivalent to one of three standard regions, and that if that Riemann surface is the universal covering surface of a compact Riemann surface of genus p, then that standard region is determined, according to whether $p=0$, $p=1$, or $p>1$, to be the Riemann sphere, the complex plane, or the unit disc. Of course, the earlier parts of the chapter are preparatory, including sections on algebraic functions (of one variable over the complex numbers), compact Riemann surfaces, and the necessary standard topological developments, namely, definition of fundamental group, genus, invariance of the genus (with respect to a certain choice of dissection of the Riemann surface), etc. This part is in general quite understandable and nice, especially the lucid reduction of topological steps to algebraic ones in handling the canonical polygon that comes from a canonical dissection of the Riemann surface. §5 on the Poisson integral is also clear and well presented. The treatment of Dirichlet's principle in this and the next section is really the climax of the chapter and is remarkable for the fact that the expression "Hilbert space" does not appear a single time. Following this comes the construction of a harmonic function with certain properties on a Riemann surface, though the statement of Theorem 2 (which seems rather obvious) is less important than some of the points introduced in the course of its proof (§8). The harmonic function so constructed can be taken as the real part of an analytic function, which will be single-valued if the Riemann surface is simply-connected. The mapping theorem (§9) and uniformization of algebraic functions (§10) are, among other things, notable for their derivation of topological facts from essentially analytical ones. All in all, it is a beautiful treatment, an excellent introduction to Riemann surfaces.

Chapter 3, the first half of volume II (there are six chapters in all, two to a volume), is a treatment of automorphic functions of one complex variable, primarily those with respect to a discontinuous group acting on the usual unit disc in the complex plane. The treatment here is based geometrically on viewing the unit disc as the non-Euclidean plane of Lobachevsky, supplied with a metric and measure which are invariant under the full group of conformal self-transformations of the unit disc (as well as under the anticonformal ones). This makes it simple to construct a fundamental region for a discontinuous group acting on the disc: One takes any point, call it x_0, not fixed by any nonidentity element of the discontinuous group, and takes as fundamental set (modulo addition of some part of its boundary) the set of points of the disc that, in the non-Euclidean sense, are nearer to x_0 than to any of its neighbors in the orbit of x_0 (under that group). In this chapter, Siegel gives a proof of one of his most interesting contributions in this particular area: Given a discontinuous group G acting on the unit disc D, such that the orbit space D/G has finite (non-Euclidean) volume, G has a fundamental region bounded by a finite number of non-Euclidean geodesics, i.e., segments of circles orthogonal to the boundary of D. He also proves that the minimum volume of such a quotient D/G is $\pi/21$, a most notable result. Poincaré series are the next subject. These are basic in the whole theory of automorphic forms (as well as in the related harmonic analysis on the group of isometries of a bounded domain). The first thing, of course, is to prove that the Poincaré series converge. At this point Siegel demonstrates the convergence of Poincaré series for one variable; in Chapter 6 he again proves the very same thing in practically identical fashion for Poincaré series of several variables, and the reviewer wonders whether this was felt essential to preserve the self-contained nature of each volume, or whether in the interests of efficiency a general proof based on general principles might not have been in order here. The next topic is the structure of the field of automorphic functions with respect to G when G acts discontinuously and freely on D and when D/G is also compact. The result is that that field is a finitely generated algebraic function field of one variable over the complex numbers. In view of the result of Siegel mentioned above, it may be remarked that something similar could also be proved when D/G has finite volume, provided some restriction is imposed on the behaviour of automorphic functions at the cusps of a fundamental domain. The reader should take note of the remark, page 60, that "it is of fundamental importance, however, that the field K can be built up using series of special type only" (i.e., series of Poincaré type having the constant 1 as the polynomial multiplying the power of the Jacobian). It may also be noted that in this part of the book there are some wrong references to section numbers and the reader is accordingly alerted. On pages 74–76 there is a very nice proof of the fact that if the universal covering surface of a compact Riemann surface is conformal to the unit disc, then the genus of the Riemann surface is greater than one. §8 of Chapter 3 deals with canonical forms of curves of genus 0 and genus 1, with attention to the j-invariant of elliptic curves. The last section, on canonical polygons, contains the striking result that given a certain type of non-Euclidean polygon in the unit disc, satisfying certain metric conditions on the lengths of its sides and the condition that the sum of all its interior angles be 2π, there is naturally associated with it a hyperbolic motion group having the interior of that polygon plus some portion of its boundary as fundamental set. This is worthy of attention because this fact has analogs in higher dimensional cases leading to the construction of non-arithmetic discontinuous groups (a topic not further discussed here). The last section concludes with the famous result that the number of automorphisms of a compact Riemann surface of genus p greater than one is at most $84(p-1)$.

The title of Chapter 4 is "Abelian integrals". The first two sections of this chapter deal with the differentials of an algebraic function field of one variable and with the existence of differentials of the first kind. The existence theorem is treated via the ideas associated with Dirichlet's principle, vol. I; however, it appears that the treatment of volume I should have been more complete, for an important point in the proof (last two lines, p. 102, vol. II) is left for the reader to fill in, using the ideas but not exactly the results of volume I. Nevertheless, the treatment is otherwise again excellent and concludes with a proof that the dimension of the linear space of differentials of the first kind is equal to the genus p. §3 is largely devoted to proving the fundamental so-called Riemann relations among the periods of integrals of the first kind on a compact Riemann surface of positive genus. Here one sees how to associate to each conformal equivalence class of compact Riemann surfaces of genus p

a complex symmetric matrix $Z=X+iY$ such that Y is positive definite. Henceforth, let $\mathfrak{H}_p$ denote the set of all p by p complex symmetric matrices Z with positive definite imaginary part; $\mathfrak{H}_p$ will be called the Siegel upper half-plane of degree p. The group $\mathrm{Sp}(p, \boldsymbol{R})$ of $2p$ by $2p$ real symplectic matrices operates on $\mathfrak{H}_p$ by generalized linear fractional transformations, and the subgroup $\Gamma=\mathrm{Sp}(p, \boldsymbol{Z})$ of integral matrices in $\mathrm{Sp}(p, \boldsymbol{R})$ (they are automatically unimodular) is called the Siegel modular group (of degree p). Then the matrix Z associated to the class of Riemann surfaces, mentioned above, is not unique, but is determined only as an element of a certain orbit of Γ in $\mathfrak{H}_p$; it might also be mentioned, aside, that if p is greater than one, not every orbit of Γ corresponds to such a class. The next sections are devoted to a preliminary treatment of the Siegel modular group and of its geometrical interpretation in connection with the canonical dissection of the Riemann surface. Leaving problems related to moduli of Riemann surfaces, the later sections of the chapter turn again to properties of Riemann surfaces as individual entities. First, the Riemann-Roch theorem for a Riemann surface is proved, entirely along classical (i.e., Hermann Weyl) lines, and of course it is also proved that a differential of the first kind has $2p-2$ zeros, and that more generally the excess of the number of zeros over the number of poles of a meromorphic differential (not identically zero) is also $2p-2$. §7 contains a proof of Abel's theorem which gives a necessary and sufficient condition in terms of integrals of differentials of the first kind for a divisor of degree zero to be a principal divisor. This section, by contrast with most parts of the book, contains some apparent inconsistencies in notation and terminology, has some rather vaguely worded passages that were difficult for this reviewer to follow, and, as if by conspiracy, is plagued by more than the usual number of omissions on the part of the proofreader; these relatively minor blemishes notwithstanding, a persistent reader will be rewarded for his efforts in this section. The remaining sections of the chapter are very good. Their subject matter, in the language of algebraic geometry, is a study of properties of the Jacobian variety of an algebraic curve of genus p. In §8, entitled "The Jacobi inversion problem", it is established (Theorem 1) that the canonically imbedded curve of genus p generates the Jacobian variety. In §9 the theta functions are introduced and in §10 the study of the zero divisors of a theta function on a curve canonically imbedded in its Jacobian variety is pursued. The main point of §§10 and 11 is investigation of "the intimate connection between the zeros of $\theta(s)$ and the explicit solution of the inversion problem". And in §11, an explicit result is given on how to express an arbitrary meromorphic function on the canonically imbedded curve C as a quotient of products of theta functions restricted to the curve. In the last section there is developed an isomorphism between the field of symmetric functions on the p-fold product of C with itself and the field of functions on its Jacobian J, or in other stronger words, an isomorphism between J and the quotient of $C\times C\times\cdots\times C$ (p factors) by the symmetric group S_p.

Volume III, comprising Chapters 5 and 6, contains much of the material appearing in an earlier set of notes entitled "Analytic functions of several complex variables", written by P. T. Bateman, and based on lectures given by Siegel at the Institute for Advanced Study some years ago. It may be said, on the one hand, that these lecture notes are more detailed in their treatment of automorphic functions of several complex variables than Chapter 6 of the current set, at least in that they give a more detailed description of the classical Cartan domains. On the other hand, the present volume contains some material that is relatively new, including an exposition of the proof of the existence of relations of algebraic dependence between automorphic forms for the Siegel modular group that relies on results of Andreotti and Grauert, utilizing the principle of pseudoconcavity, that post-date the aforesaid lecture notes.

Chapter 5, entitled "Abelian functions", commences with the development of some standard facts about several complex variables. These include the Weierstrass preparation theorem for power series (of which an interesting and, to the reviewer, novel proof is given), Cauchy's integral formulas in the simplest standard form, meromorphic functions, local divisors, and the solution of Cousin's problem for C^n (the space of n complex variables), showing the existence of entire functions with prescribed integral divisors (if $n=1$, this is essentially due to Weierstrass). This in turn affords the existence of meromorphic functions with prescribed divisors, and shows specifically that a meromorphic function in C^n can be written as the quotient of entire functions which have everywhere relatively prime divisors. Now we assume that a lattice Ω of periods is given in C^n, along with a divisor $\mathfrak{d}$ such that $\mathfrak{d}$ is stable with respect to translations by all elements of Ω. Then $\mathfrak{d}$ may be written as $\mathfrak{d}_1-\mathfrak{d}_2$, where $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are integral divisors without common component, and $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are individually invariant under the action of Ω. If there exists a meromorphic function f (not identically zero) with the divisor $\mathfrak{d}$ such that $f(z+a)\equiv f(z)$ for each $a\in\Omega$, then f may be expressed as the quotient of two entire functions, $f=g/h$, such that the divisor of g is $\mathfrak{d}_1$ and that of h is $\mathfrak{d}_2$. This implies that g (or h) satisfies a condition of the form $g(z+a)\equiv e^{w_a(z)}g(z)$, where $w_a(z)$ is an entire function depending on the element $a\in\Omega$. Such a function g is called a Jacobian function, while f is called an Abelian function. The elements of Ω are called periods of f (or quasi-periods of g and h). If f has no other periods (respectively, if g has no other quasi-periods) than those in G, then f (respectively, g) is called nondegenerate.

Let C be a period matrix for Ω (in Chapter 4, the period matrix is denoted by Q). If a nondegenerate Abelian function (or nondegenerate Jacobian function) exists for Ω, then C satisfies the conditions bearing the name of Riemann, of which a special case was already discussed in Chapter 4 (vol. II, pp. 112–113), and the general case of which is obtained in replacing the skew-symmetric matrix J of Chapter 4 by an arbitrary rational, skew-symmetric matrix. Conversely, the Riemann conditions are also sufficient for the existence of nondegenerate Abelian and Jacobian functions. In particular, one has the result that a periodic divisor is the divisor of a Jacobian function. As for the exponents $w_a(z)$, by the establishment of certain inequalities stemming from the application of Cousin's lemma to a decomposition of C^n into cubes, and by a version of Schwarz' lemma, one shows they may be assumed to be polynomials. Then, by solving a system of difference equations, one may take them to be linear polynomials. All of this is standard, but Siegel's treatment is very clear, and well recommended to one who wants to learn the subject. The last part of this chapter is devoted to proving that the orbit space C^n/Ω, with its standard complex structure, may be realized as a projective algebraic variety, provided the Riemann conditions are satisfied, in other words, if there exists a nondegenerate meromorphic function with precisely the period lattice Ω. Moreover, this orbit space, or torus, may *then* be imbedded in some complex projective space, using as the projective coordinates of the imbedding a basis of equivariant Jacobian functions of suitable exponent system w_a. In this section again the proofreading is deficient and the reader should be alert. The chapter ends with a formulation of the addition theorem for Abelian functions (which amounts essentially to an algebraic formulation of the group law on the Abelian (or, as Siegel calls it, Picard) variety C^n/Ω), and with a proof that any connected, complex, projective group variety is an Abelian variety.

The last chapter is on the general subject of automorphic functions of several complex variables. First, the general notions of a discontinuous group Γ acting on a domain D in C^n and of automorphic form and function with respect to Γ are introduced. Then it is shown how to construct automorphic forms, at least when D is bounded, by means of Poincaré series, the convergence of which is proved over again, *mutatis mutandis* from the one variable case. Siegel then proceeds to the classical Cartan domains, of which the main example discussed is that of the generalized unit disc $\mathfrak{E}_n$ of symmetric n by n complex matrices Z such that $E_n - \bar{Z}Z > 0$. It is equivalent via the Cayley transformation to the Siegel upper half-plane $\mathfrak{H}_n$ of degree n. The symplectic group $\mathrm{Sp}(n, \boldsymbol{R})$ may then be viewed as operating on either domain. From this comes the introduction of symplectic geometry, including an Hermitian metric and volume element or measure on D, both invariant under $\mathrm{Sp}(n, \boldsymbol{R})$; the existence of the invariant metric makes it possible to construct the fundamental domain of a discontinuous group Γ operating on D in purely geometric fashion, in the same manner as indicated before. If one assumes that the quotient D/Γ is compact, then one may prove in more or less elementary fashion that the field of meromorphic functions on D invariant under Γ (automorphic functions with respect to Γ) is an algebraic function field of the appropriate dimension. An obvious example of a discontinuous group acting on $\mathfrak{H}_n$ is the arithmetic group $\Gamma=\mathrm{Sp}(n, \boldsymbol{Z})$ mentioned earlier. In this case, a fundamental domain will not be relatively compact in $\mathfrak{H}_n$, and Siegel here reproduces his earlier (1939) construction of a fundamental domain for Γ. In order to develop the connection of this situation with moduli of Abelian varieties, Siegel now comes back to the subject of Abelian functions. Among other standard facts proven here is that a holomorphic mapping of one Abelian variety into another is the sum of a homomorphism (of complex Lie groups) and of a constant translation, and this fact is used in the discussion of the multiplier algebra of the period matrix of an Abelian variety. Furthermore, the classical relationship between the j-function and Eisenstein series for the case $n=1$ and the moduli of elliptic curves is developed to provide a model for the higher dimensional case, wherein there exists a natural one-to-one correspondence between the orbits of the so-called inhomogeneous modular group of level T and isomorphism classes of those Abelian varieties whose period matrices *can be and are* taken in the standardized form $(T\,Z)$, where T is an integral n by n diagonal matrix in standard elementary divisor form and $Z \in \mathfrak{H}_n$ (the phrase "can be and are taken" is a rough way of expressing the notion of polarization which there is no room to explain fully here). The point is that the orbit space in this case is essentially the moduli space of Abelian varieities with period matrix so normalized. However, Siegel does not elaborate on this point. Complex multiplication is touched on here with reference to the example at the beginning of Chapter 1, but the discussion is very brief. It would have been convenient for the reader to have had some references at this point to the extensive bibliography at the end of volume III, in case he wanted to pursue the matter. The rest of the book concerns the Siegel modular group with fundamental domain constructed in analogy with the well-known case $n=1$. In the case $n=1$, we have two Eisenstein series which generate the whole algebra of modular forms. Much less is known about the Eisenstein series on $\mathfrak{H}_n$, $n>1$, which are introduced at this point. At any rate their convergence is demonstrated, and the lines of the proof are very similar to those of Godement's proof for the convergence of Eisenstein series for an algebraic group; the reviewer regrets that he cannot fully inform the reader concerning the historical connection here. At any rate, these Eisenstein series afford another way of constructing automorphic forms on $\mathfrak{H}_n$ with respect to Γ, and are in a certain technical sense complementary to the Poincaré series constructed earlier. Siegel concludes this final chapter with investigation of the field of modular functions (read "automorphic functions with respect to Γ"). He first shows that that *field* at least *is* generated by the Eisenstein series (very little is known about generation of the ring or algebra of modular forms except for Igusa's results for very small n), and then establishes that it is an algebraic function field of finite degree over a purely transcendental extension of C of degree $n(n+1)/2$. The proof of the former fact is Siegel's own (Math. Ann., 1939), and although Siegel established the second fact in the same paper, the proof of it here is due to Andreotti and Grauert, who base their proof on a property of pseudoconcavity connected with the modular group.

Although minor flaws, such as the scarcity of references to the extensive bibliography when there is not space to pursue a subject, and the laxness of the proofreader(s) exist, this reviewer's opinion is that the good far outweighs these trivial shortcomings. These are small blemishes, and the work as a whole stands out for its excellent treatment of a broad subject, and what, for obvious reasons of space, it lacks in completeness, it more than makes up for in inspirational quality.

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Algebraic graph theory, by Norman Biggs, Cambridge Tracts in Mathematics No. 67, Cambridge University Press, 1974, vii+170 pp., $11.70

Combinatorial theory seminar, by Jacobus H. van Lint, Eindhoven University of Technology, Lecture Notes in Mathematics No. 382, Springer-Verlag, 1974, vi+131 pp., DM 18

Some of the most satisfying and fruitful developments in mathematics have occurred when bridges have been discovered between seemingly disparate branches of the subject. Then the results and methods of the one branch have become applicable to the other, and at best there has been an equal flow in the reverse direction also. Thus the Zeta function of Riemann allowed complex function theory to illuminate the theory of the distribution of prime numbers, and thereby the theory of entire functions was stimulated also.

Prerequisites, for maximum impact in such situations, are naturalness