ERRATA (Communicated by the author)

		Should read
p. 32	line 3	$\boldsymbol{R}$-rank G
p. 49	line 4	$\geqq$
p. 69	line 3	$\Gamma\backslash X$
p. 134	line 10	$\boldsymbol{Sp}(1,n)/\boldsymbol{Sp}(1)\times\boldsymbol{Sp}(n)$
p. 136	line −7	projective space P_K^n as cyclic $\boldsymbol{K}$-subspaces of $\boldsymbol{K}^{n+1}$, represent them as hermitian projections onto cyclic $\boldsymbol{K}$-subspaces of $\boldsymbol{K}^{n+1}$ with respect to
p. 142	line 2	H_K^n
p. 187	line 5	$\cdots\theta$ extends to a unique analytic isomorphism $\cdots$
p. 187	line 6	$\cdots$ from Theorem 18.1, Corollary 23.6, and Lemma 8.6.

SIGURDUR HELGASON

Complete normed algebras, by F. F. Bonsall and J. Duncan, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 80, Springer-Verlag, New York, Heidelberg, Berlin 1973, x+301 pp. $26.20

It was in 1939 that I. M. Gelfand [**10**] announced the results of his pioneering investigations of Normed Rings, thereby launching a new field of mathematical research which continues 35 years later in a state of vigorous development. For Gelfand, a normed ring was in fact a complete normed algebra; i.e., an algebra for which the underlying vector space is a (usually complex) Banach space and multiplication is continuous with respect to the given Banach space norm. Continuity of multiplication is usually provided by imposing the multiplicative inequality, $\|xy\|\leqq\|x\|\,\|y\|$, on the norm. For obvious reasons, these algebras have come to be known as "Banach algebras", a term which is now rather firmly established in the literature.[1] Such algebras were in fact studied earlier by M. Nagumo [**18**] and K. Yosida [**26**] who called them "metric rings". Also, as might be expected, some of the concepts arising in the earlier study of operators on a Banach space, as well as the study of certain

[1] The authors remark (p. 4) that they would have preferred the term "Gelfand algebra" for a complete normed algebra. Although the reviewer had much to do with establishing the term "Banach algebra" and has a strong preference for terminology that suggests the nature of the indicated object, he agrees that "Gelfand algebra" would have been a most appropriate choice. Since this book will no doubt be widely accepted, the authors, given the courage of their convictions, probably could have effected the change.

special Banach spaces, are closely related to Banach algebra notions. In addition we have the highly developed theory of Rings of Operators, now called von Neumann algebras, a subject which dates back to a 1929 paper of J. von Neumann [19] and subsequently developed in a series of papers by F. J. Murray and von Neumann appearing between 1935 and 1943. Since these algebras are also Banach algebras, there has naturally been substantial interaction between the two theories. However, the study of von Neumann algebras is dependent on special techniques so peculiar to these algebras that the subject continues to grow more or less independently of the general theory.

In spite of these earlier studies, it was Gelfand who gave the subject its proper setting through his recognition of the fundamental roles played by elementary ideal theory and an elegant characterization of the complex numbers as a normed division algebra. A proof of this characterization, announced without proof by S. Mazur [17], was provided by Gelfand [11]. His fundamental result was that every commutative Banach algebra (with unit) is homomorphic to an algebra of continuous functions on a certain compact Hausdorff space (maximal ideal space), the kernel of the homomorphism being the radical (intersection of maximal ideals) of the algebra. The key fact here is that the algebra, modulo a maximal ideal, is isomorphic with the complex numbers. Gelfand also applied his theory to a number of algebras of great interest in Analysis, exhibiting an equivalence of natural Banach algebra concepts with important analysis concepts. These applications provided the motivation, necessary at the time, for analysts to adopt the algebra approach[2] required by the Gelfand theory.

It is remarkable that in a series of papers, published between 1939 and 1944 by Gelfand and his collaborators, virtually all of the main lines along which the theory of Banach algebras would develop for a period of 25 or 30 years were already laid down. The only important exception was the involvement with the theory of analytic functions of several complex variables (SCV), initiated in 1953 by a paper of G. E. Shilov [22] in which an operational calculus for several Banach algebra elements was introduced. A limitation in Shilov's result, which amounted to a restriction to finitely generated algebras, was removed by R. Arens and A. P. Calderón [4] and the end result became known as the Shilov-Arens-Calderón theorem. In 1954 and presumably independently of Shilov, L. Waelbroeck [25] gave a more general and, as it turns out, a more

[2] Although a "linear space" point of view had found its way into Analysis *via* Functional Analysis, an analogous "algebra" point of view was much later in coming. An operation of multiplication, when available, tended to be regarded as a convenient tool rather than as a structure property.

natural definition of the operational calculus. The Waelbroeck approach was unfortunately neglected by many (including the reviewer) perhaps because Shilov's approach was through A. Weil's SCV generalization of the Cauchy integral formula and therefore related directly to the familiar Cauchy formula method of defining functions of single elements. In any case, the bringing of SCV methods into the study of Banach algebras has turned out to be a stimulating and productive event. It has also led to a number of applications of Banach algebra methods in SCV.

The field of Banach algebras came into existence through a bringing together of ideas from Analysis and Algebra and the interaction of the two disciplines continues to be an important stimulus to the development of the subject. Although sharp distinctions between the traditional fields of mathematics no longer exist, there is nevertheless a difference in approach within Analysis and Algebra which is clearly reflected in the work on Banach algebras. Roughly speaking, the Analysis approach tends to concentrate on certain special concrete examples, or their generalizations, and involves extensive use and generalization of results from classical analysis, with algebra often playing a secondary role. On this side we have, for example, the study of group algebras (as a setting for abstract harmonic analysis) and function algebras, both of which have received much attention over the years. The study of function algebras has been especially active in the last 15 years or so, involving progressively deeper results from "hard" analysis. As might be expected, the bulk of the function algebra problems are concerned in one way or another with analyticity questions going back either to one or to several complex variables.

The Algebra approach is dominant in the "general theory" of Banach algebras, where the problems tend to be concerned with structure and representation theory while the Analysis involved is generally Functional Analysis. In this area we have some very interesting and subtle interactions of the two fields. An especially good example is the beautiful result, finally proved by B. E. Johnson [**13**], that the topology of a semisimple Banach algebra is uniquely determined, i.e. any two norms under which it is a Banach algebra must be equivalent. Numerous other such results could be cited, including the many interesting topological overtones that occur in the Banach algebra versions of the standard theory for rings.

The opening paragraph of the present book provides an unusually clear statement of the nature and importance of the theory of Banach algebras in mathematics as well as an indication of the spirit in which the book is written:

> The axioms of a complex Banach algebra were very happily chosen. They are simple enough to allow wide ranging fields of application, notably in harmonic analysis, operator

> theory, and function algebras. At the same time they are tight enough to allow the development of a rich collection of results, mainly through the interplay of the elementary parts of the theories of analytic functions, rings and Banach spaces. Many of the theorems are things of great beauty, simple in statement, surprising in content, and elegant in proof. We believe that some of them deserve to be known by every mathematician.

The authors' aim is "to give an account of the principal methods and results in the theory of Banach algebras, both commutative and non-commutative." However, certain of the special classes of algebras, such as C^*-algebras, function algebras, and group algebras are not treated in detail. Also omitted are a few topics that might naturally have been included such as the theory of multipliers, extensions of Banach algebras and the implications for Banach algebras of some of the standard conditions on rings; and, finally, the various generalizations of Banach algebras are not included. The emphasis is clearly on the general theory with the algebra approach much in evidence, especially in the later chapters. Therefore the book might be regarded as a sequel to the reviewer's book on the *General theory of Banach algebras* [**21**] which appeared in 1960. In the intervening period, there has been much progress, so an updating of the subject was overdue. Also there has been a recent upsurge of interest in the general theory stimulated in part by some of the development in ring theory. Therefore this book makes its appearance at an opportune time. It is well organized and well written with numerous cross-references which make for easy reading. The style is somewhat formal but is periodically relieved by examples and remarks directed to the literature as well as various open questions. The proofs, in many cases quite elegant, are carefully and clearly, though sometimes rather cryptically, presented. All in all, this is a fine book which will be very useful both to the specialist and also to anyone else who might want an introduction to the subject.

The book is divided into 50 sections which are in turn grouped into seven chapters. The chapter titles and section headings give in most instances a fair indication of the contents. However we have included a few remarks to bring out some of the special features.

Chapter I. *Concepts and elementary results.* (§1. Normed algebras. §2. Inverses. §3. Quasi-inverses. §4. Equivalent norms. §5. The spectrum of an element of a complex normed algebra. §6. Contour integrals. §7. A functional calculus for a single Banach algebra element. §8. Elementary functions. §9. Ideals and modules. §10. The numerical range of an element of a Banach algebra. §11. Approximate identities. §12. Involutions.

§13. The complexification of a real algebra. §14. Normed division algebras.)

In §5 a simple abstract Runge theorem is proved for elements of a Banach algebra and applied in §7 to give a nice proof of the classical Runge theorem on rational approximation of holomorphic functions. A review of the elementary theory of integration of vector-valued functions is given in §6 as a basis for the functional calculus in §7. This section also contains an elegant proof of the Cauchy formula for complex functions defined on a "punched disc" (i.e. a closed disc minus an arbitrary finite union of open discs) and a generalization of the Taylor and Laurent series expansions of holomorphic functions which provides a simple treatment of the functional calculus in §7. The numerical range of an element a of a Banach algebra A (with unit) is the set $V(a)=\{f(a):f\in A', \|f\|=f(1)=1\}$, where A' denotes the Banach space dual of A. It is a convex, compact set that contains the spectrum of a. §10 contains some of the properties of numerical range needed later in the discussion of star algebras. The main results in §11 are concerned with the factoring of elements of a Banach algebra due essentially to P. Cohen [9]. §14 contains, in addition to the Mazur-Gelfand theorem, a complete proof that every normed division algebra over the real numbers is isomorphic to either the reals, complexes or quaternions.

Chapter II. *Commutativity.* (§15. Commutative subsets. §16. Multiplicative linear functionals. §17. The Gelfand representation of a commutative Banach algebra. §18. Derivations and automorphisms. §19. Generators and joint spectra. §20. A functional calculus for several Banach algebra elements. §21. Functions analytic on a neighborhood of the Carrier space. §22. The Shilov boundary. §23. The Hull-Kernel topology.)

In §18 a number of results for derivations on a Banach algebra to itself and the connection with automorphisms of the algebra are obtained. Included here is the result due to I. Singer and J. Wermer [23] that every continuous derivation on a complex commutative Banach algebra A maps A into the radical and the result due to B. E. Johnson [14] that if A is semisimple, then the only derivation (continuous or not) is zero. The Shilov-Arens-Calderón theorem, mentioned earlier, is proved in §§19 and 20. In §21 the carrier space Φ_A (space of maximal ideals) is regarded as a subset of A', the Banach space dual of A, where A' is given the weak* topology. A function f defined on an open neighborhood of Φ_A in A' is called "analytic" if it is locally bounded and each function of the form $f\circ\pi$ is holomorphic in the usual sense on $\pi^{-1}(D)$, where $\pi:C^n\to A'$ is an affine map and n is arbitrary. The main result, due to G. R. Allan [2] asserts that there exists $u\in A$ such that $f(\varphi)=\varphi(u)$ for $\varphi\in\Phi_A$.

Chapter III. *Representation theory.* (§24. Algebraic preliminaries. §25. Irreducible representations of Banach algebras. §26. The structure space of an algebra. §27. A-module pairings. §28. The dual-module of a Banach algebra. §29. The representation of linear functionals.)

Use is frequently made of the equivalence of representations of A on a linear space with left A-modules. In §25 it is proved that every representation of a Banach algebra A in the bounded operators on a normed linear space is automatically continuous. This is essentially B. E. Johnson's uniqueness of norm theorem mentioned earlier. In §26 we have the representation of a semisimple Banach algebra as a normed subdirect sum of primitive Banach algebras. The structure space is the space of primitive ideals. The last three sections (§§27–29) contain results of the authors **[6]**, **[7]** in which some of the linear functional techniques, so useful in the case of commutative and star algebras, are extended to the general case.

Chapter IV. *Minimal ideals.* (§30. Algebraic preliminaries. §31. Minimal ideals in complex Banach algebras. §32. Annihilator algebras. §33. Compact action on Banach algebras. §34. H^*-algebras.)

In this chapter the sharper structure theorems available when the algebras contain minimal ideals are obtained. The results are especially nice for annihilator algebras which are defined by the condition that the right (left) annihilator of a closed left (right) ideal is equal to zero if and only if the ideal is the whole algebra. The concept is due to Bonsall and A. W. Goldie **[8]**. §32 contains an elegant treatment of these algebras. In §33 a Banach algebra A is defined to be *compact* if the mapping, $a \mapsto tat$, for each $t \in A$ defines a compact linear operator on A. This notion is due to J. C. Alexander **[1]**. If A is compact, then each of its primitive components is compact and admits a norm reducing isomorphism with an irreducible algebra of compact operators on a Banach space. In §34 an H^*-algebra is defined to be a Banach star algebra whose norm is given by an inner product, (x, y), such that $(ax, y) = (x, a^*y)$ and $(xa, y) = (x, ya^*)$ for all a, x, y in the algebra. H^*-algebras were introduced by W. Ambrose **[3]** who did not require a unique a^* for each a. The notions coincide, however, for semisimple algebras.

Chapter V. *Star algebras.* (§35. Commutative Banach star algebras. §36. Continuity of the involution. §37. Star representations and positive functionals. §38. Characterizations of C^*-algebras. §39. B^*-seminorms. §40. Topologically irreducible star representations. §41. Hermitian algebras.)

Aside from function algebras, the most studied Banach algebras are those possessing an involution. These algebras (called here Banach star algebras), which include C^*-algebras (closed selfadjoint algebras of operators on

Hilbert space) and hence von Neumann algebras, provide one of the richest and most satisfying portions of the general theory. This chapter contains refinements and extensions of some of the well-known results for star algebras and their representations. Properties of the numerical range of an element of a Banach algebra are used in §38 to give an elegant treatment of the characterization of C^*-algebras as B^*-algebras. This section also contains an interesting characterization of B^*-algebras in terms of numerical range attributed to I. Vidav [24] and T. W. Palmer [20].

Chapter VI. *Cohomology.* (§42. Tensor products. §43. Amenable Banach algebras. §44. Cohomology of Banach algebras.)

A brief introduction to the cohomology groups, $H^n(A, X)$, of a Banach algebra A with coefficients in a Banach A-bimodule X is given in §44. The group $H^1(A, X)$ is equal to the linear space of all bounded X-derivations of A (i.e. bounded linear transformations $D: A \to X$ such that $D(ab) = aDb + (Da)b$) modulo its subspace of inner X-derivations (i.e. the derivations $\delta_x: a \mapsto ax - xa$, $x \in X$). Thus $H^1(A, X) = (0)$ means that every bounded X-derivation is inner. Associated with the A-bimodule X we have the dual A-bimodule X', where X' is the dual of X and, for $a \in X$ and $x' \in X'$, $(ax')(x) = x'(xa)$ and $(x'a)(x) = x'(ax)$, $x \in X$. If $H^1(A, X') = (0)$ for every X, then the algebra A is said to be *amenable.* The terminology comes from the fact that a group G is amenable (admits an invariant mean) if and only if the discrete group algebra $l^1(G)$ is amenable. §43 contains a number of results for amenable Banach algebras due mainly to B. E. Johnson [15], [16]. Some of the proofs involve tensor products discussed in §42.

Chapter VII. *Miscellany.* (§45. Quasi-algebraic elements and capacity. §46. Nilpotents and quasi-nilpotents. §47. Positiveness of the spectrum. §48. Type 0 semialgebras. §49. Locally compact semialgebras. §50. Q-algebras.)

A definition of *capacity* for elements in a Banach algebra is given in §45. The capacity of an element turns out to be equal to the capacity, in the classical sense, of its spectrum. An element is *quasi-algebraic* if its capacity is zero. The latter notion is due to P. R. Halmos [12]. §§47–49 contain topics from the theory of semialgebras which was initiated by the first author [5]. In §50, a *Q-algebra* (*IQ-algebra*) is a complex commutative Banach algebra A which is bicontinuously (isometrically) isomorphic to a Banach algebra of the form B/J, where B is a closed subalgebra of $C(X)$ for some compact Hausdorff space X and J is a closed ideal in B. The concept is due to N. Th. Varopoulos. It is not very clear why the authors choose to include Chapter VII in its present form. All of the material except for semialgebras, a topic which might conceivably have rated a more complete treatment in a chapter of its own, seemingly could have been incorporated into the other chapters.

The Bibliography, which is not intended to be comprehensive, consists mainly of items that relate to the general theory of Banach algebras and no attempt was made to cover areas such as function algebras, C^*-algebras, von Neumann algebras, harmonic analysis, numerical range, and general topological algebras. The fact that it nevertheless contains 488 items gives some indication of the amount of activity that there has been in this area. There is also a very good general index plus a useful special index of symbols.

C. E. RICKART

REFERENCES

1. J. C. Alexander, *Compact Banach algebras*, Proc. London Math. Soc. (3) **18** (1968), 1–18. MR **37** #4618.

2. G. R. Allan, *An extension of the Šilov-Arens-Calderón theorem*, J. London Math. Soc. **44** (1969), 595–601. MR **39** #4678.

3. W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc. **57** (1945), 364–386. MR **7**, 126.

4. R. Arens and A. P. Calderón, *Analytic functions of several Banach algebra elements*, Ann. of Math. (2) **62** (1955), 204–216. MR **17**, 177.

5. F. F. Bonsall, *Semi-algebras of continuous functions*, Proc. London Math. Soc. (3) **10** (1960), 122–140; *ibid* (3) **12** (1962), 133–143; (3) **13** (1963), 51–70. MR **22** #2892; **24** #A2862; **26** #615.

6. F. F. Bonsall and J. Duncan, *Dual representations of Banach algebras*, Acta Math. **117** (1967), 79–102. MR **34** #4929.

7. ———, *Dually irreducible representations of Banach algebras*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 97–111. MR **38** #535.

8. F. F. Bonsall and A. W. Goldie, *Annihilator algebras*, Proc. London Math. Soc. (3) **4** (1954), 154–167. MR **15**, 881.

9. P. J. Cohen, *Factorization in group algebras*, Duke Math. J. **26** (1959), 199–205. MR **21** #3729.

10. I. M. Gelfand, *To the theory of normed rings*, Dokl. Akad. Nauk. SSSR **23** (1939), 430–432; **25** (1939), 570–572, 573–574. MR **1**, 330; **1**, 331.

11. ———, *Normierte Ringe*, Mat. Sb. **9** (1941), 3–24.

12. P. R. Halmos, *Capacity in Banach algebras*, Indiana Univ. Math. J. **20** (1971), 855–863. MR **42** #3569.

13. B. E. Johnson, *The uniqueness of the (complete) norm topology*, Bull. Amer. Math. Soc. **73** (1967), 537–539. MR **35** #2142.

14. ———, *Continuity of derivations on commutative algebras*, Amer. J. Math. **91** (1969), 1–10. MR **39** #7433.

15. ———, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127** (1972).

16. ———, *Approximate diagonals and cohomology of certain annihilator Banach algebras*, Amer. J. Math. **94** (1972), 685–698.

17. S. Mazur, *Sur les anneaux linéaires*, C. R. Acad. Sci. Paris **207** (1938), 1025–1027.

18. M. Nagumo, *Einige analytische Untersuchungen in linearen metrischen Ringen*, Japan J. Math. **13** (1936), 61–80.

19. J. von Neumann, *Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren*, Math. Ann. **102** (1929), 370–427.

20. T. W. Palmer, *Characterizations of C^*-algebras*, Bull. Amer. Math. Soc. **74** (1968), 538–540. MR **36** #5709.

21. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, University Series in Higher Math., Princeton, N.J., Reprinted (with corrections) by Robert Krieger Publ. Co., 1974. MR **22** #5903.

22. G. E. Šilov, *On the decomposition of a commutative normed ring in a direct sum of ideals*, Mat. Sb. **32** (74) (1953), 353–364; English transl., Amer. Math. Soc. Transl. (2) **1** (1955), 37–48. MR **14**, 884; **17**, 512.

23. I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. **129** (1955), 260–264. MR **16**, 1125.

24. I. Vidav, *Eine metrische Kennzeichnung der selbstadjungierten Operatoren*, Math. Z. **66** (1956), 121–128. MR **18**, 912.

25. L. Waelbroeck, *Le calcul symbolique dans les algèbres commutatives*, J. Math. Pures Appl. (9) **33** (1954), 147–186. MR **17**, 513.

26. K. Yosida, *On the group embedded in the metrical complete ring*. Japan. J. Math. **13** (1936), 459–472.

The structure of factors, by S. Anastasio and P. M. Willig, Algorithmics Press, New York, 1974, iii+116 pp.

In a paper appearing in the 1929 Mathematische Annalen (*Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren*), von Neumann initiated the study of *Rings of operators* (renamed *von Neumann algebras* in J. Dixmier's classic, *Les algèbres d'opérateurs dans l'éspace Hilbertien*, Paris, 1957). These are algebras, R, of bounded linear transformations (operators) of a Hilbert space H into itself, closed in the strong-operator topology ($A_n \rightarrow A$ means that $A_n x \rightarrow Ax$, for each x in H) and having the property that A^*, the adjoint of A, is in R if A is. Von Neumann saw two motivating forces behind the study of these algebras: applications to the newly emergent Quantum Physics, and application to the study of infinite groups. Quantum Physics, as it was being formulated, was involved with algebraic combinations of (selfadjoint) operators. It was certain to require (at the mathematical level) a deeper understanding of the structure of algebras of operators. The technique of group algebras had been so useful in the study of finite groups that some corresponding construct for infinite groups was certain to be crucial for their analysis.

The detailed study of von Neumann algebras was undertaken in a series of papers written in collaboration with F. J. Murray. The first appeared in the 1936 Annals of Mathematics, *On rings of operators*. Since noncommutativity was the basic technical problem, Murray and von Neumann moved quickly to the study of those von Neumann algebras, *factors*, whose centers consist of scalar multiples of the unit element.

As in much of Functional Analysis, the statements of results in the theory of operator algebras are algebraic in flavor. The ideas, proofs and