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THE q-REGULARITY OF LATTICE POINT PATHS IN R^n

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1. **Introduction.** Given any set X and a cardinal number q, then, following Rado [4], a collection $\mathcal{S}$ of sets is called q-regular in X if, whenever X is partitioned into q parts, then at least one part contains as a subset a member of $\mathcal{S}$. More generally, by requiring the partitions of X to belong to a given preassigned family $\mathcal{F}$, we obtain the notion of q-regularity in X *relative to* $\mathcal{F}$. Letting N denote the positive integers, given $q \in N$, it is convenient to regard a partition of a set X into q parts as a function f: $X \to Z_q$, where Z_q denotes the ring of integers modulo q. The partition $\mathcal{P}(f) = \{f^{-1}(\overline{m})\colon m \in N\}$ of X is said to be represented by f, where $\overline{m}$ denotes the residue class in Z_q containing $m \in N$. Given f: $X \to Z_q$, g: $Y \to Z_q$, then, as in [2], we obtain a partition $f \oplus g$: $X \times Y \to Z_q$ by the formula $(f \oplus g)(x, y) = f(x) + g(y)$, where the sum on the right takes place in Z_q. If $A \subseteq R^n$ is a subset of euclidean n-space R^n, let $\mathcal{F}^{\oplus}(A)$ denote the family consisting of those partitions of A which are representable by functions $(f_1 \oplus \cdots \oplus f_n)|A$: $A \to Z_q$, where f_i: $R \to Z_q$, $i = 1, \cdots, n$, and where $g|A$ denotes the restriction of the function g to A.

A (linear) *lattice point path* in R^n shall mean the intersection of a connected subset of a straight line in R^n with the lattice points $Z^n \subset R^n$, where Z denotes the set of integers. Adding a maximal element ∞ to R, and given any $j \in N^* = N \cup \{\infty\}$, let $\mathcal{L}_j$ denote the collection of lattice point paths obtainable from lines which have a set of integer direction numbers bounded in absolute value by j, and let $\mathcal{S}_{j,k} \subset \mathcal{L}_j$ denote the subcollection of $\mathcal{L}_j$ consisting of those paths which contain k points, $k \in N^*$.

Given any $A \subseteq R^n$, $j \in N^*$, and $q \in N$, we then define

$$\rho_{j,q}(A) = \sup\{k \in N\colon\ \mathcal{S}_{j,k} \text{ is } q\text{-regular in } A\},$$

$$\rho_{j,q}^{\oplus}(A) = \sup\{k \in N\colon\ \mathcal{S}_{j,k} \text{ is } q\text{-regular in } A \text{ relative to } \mathcal{F}^{\oplus}(A)\},$$

where we set $\rho_{j,q}(A) = \rho_{j,q}^{\oplus}(A) = 0$ if $A \cap Z^n = \varnothing$.

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Note that the functions $\rho_{j,q}, \rho^{\oplus}_{j,q}$ are monotone, and that $\rho_{j,q}(A) \leqslant \rho^{\oplus}_{j,q}(A)$ for all $A \subseteq R^n$. The case $j = 1, q = 2$ is of special interest, so that we then suppress the subscripts, writing $\rho = \rho_{1,2}, \rho^{\oplus} = \rho^{\oplus}_{1,2}$. For example, one of our main results is the formula $\rho^{\oplus}(Z^n) = n, n \in N$, where we conjecture that this formula also holds when $\rho^{\oplus}$ is replaced by ρ (it *does* hold for ρ when $n \leqslant 3$). Also, letting $C^n(m)$ denote an n-dimensional hypercube of lattice points having m points on a side, $C^n(m) = \{(x_1, \cdots, x_n) \in Z^n\colon\ 1 \leqslant x_i \leqslant m, i = 1, \cdots, n\}$, note that $\rho(C^n(m)) \leqslant m$, where the equality $\rho(C^n(m)) = m$ can be interpreted to imply that n-dimensional Tic-Tac-Toe *cannot* be played to a tie in $C^n(m)$ (where a winning set in $C^n(m)$ consists of m points in a straight line). The following proposition has a simple verification.

PROPOSITION 1. $\rho^{\oplus}(C^n(n)) = n, n \in N$.

2. **Statement of main results.** The following three theorems represent our main results on ρ, $\rho^{\oplus}$, and $\rho^{\oplus}_{\infty,2}$.

THEOREM 1. $\rho^{\oplus}(Z^n) = n, n \in N$.

THEOREM 2. $\rho(C^n(n)) \leqslant n - 1, n \geqslant 4$.

THEOREM 3. $\rho^{\oplus}_{\infty,2}(Z^n) \leqslant 2n - 1, n \in N$.

REMARKS. 1. Theorem 2 is surprising in view of the contrasting fact that $\rho(C^n(n)) = n, n \leqslant 3$ (compare also with Proposition 1). Hales and Jewett have shown [2, Theorem 5] that the winning sets in $C^n(n + 1)$ are not 2-regular in $C^n(n + 1), n \in N$, i.e., in our terminology, $\rho(C^n(n + 1)) \leqslant n$. They actually show (again in our terminology) that $\rho^{\oplus}(C^n(n + 1)) \leqslant n$, although it turns out that the partitions they use *cannot* be extended to partitions of Z^n satisfying the requirements of Theorem 1. Note that the result $\rho(C^n(n + 1)) \leqslant n$ also follows immediately from Theorem 1, while Theorem 2 improves this latter result in the dimensions $n \geqslant 4$. Even in case the winning sets in $C^n(m)$ are not 2-regular in $C^n(m)$, it still might not be possible for the second player to force a tie. For results on when the second player *can* force a tie, see [1] and [2].

2. To obtain a function dependent upon $\rho_{j,q}$, but which, unlike $\rho_{j,q}$, is invariant under affine isomorphisms of R^n, we define, for $A \subseteq R^n$,

$$\lambda_{j,q} = \sup\{\rho_{j,q}(f(A))\colon\ f\colon R^n \to R^n \text{ is an affine map}\},$$

with $\lambda^{\oplus}_{j,q}$ defined similarly using $\rho^{\oplus}_{j,q}$ in place of $\rho_{j,q}$. Setting $\lambda = \lambda_{1,2}, \lambda^{\oplus} = \lambda^{\oplus}_{1,2}$, we see from Proposition 1 and Theorem 1 that $\lambda^{\oplus}$ distinguishes in a natural geometric-combinatorial way amongst the various euclidean spaces. Indeed, we have the following corollary, which we conjecture also holds for λ.

COROLLARY 1. *$\lambda^{\oplus}(U) = n$, whenever U is a nonempty open set in R^n, $n \in N$.*

3. **Description of the partitions used in our main results.** Given $m \in N$, let τ_m: $Z \to Z$, ϕ_m: $Z \to Z$ be defined by $\tau_m(x) = x + m$, $\phi_m(x) = [x/m]$, where $[y]$ denotes the greatest integer $\leqslant y$. Theorems 1, 2, and 3 depend on a remarkable sequence $\{g_n\}$ of functions from Z into Z_2 defined by the formulas

I. $g_1(x) = \bar{x} \quad (x \in Z)$,

II. $g_{2m} = g_1 \circ \phi_{2m} \quad (m \in N)$,

III. $g_{n-1} = g_n + g_n \circ \tau_1 \quad (n \geqslant 2)$.

These are overdefinitions, but turn out to be consistent. Theorem 1 is verified using the function $g_1 \oplus \cdots \oplus g_n$: $Z^n \to Z_2$, while Theorem 3 is verified using $g_n \oplus \cdots \oplus g_{2n-1}$: $Z^n \to Z_2$. Theorem 2 is verifed by the restriction, to a suitable translate of $C^n(n)$, of $f \oplus g_1$, when $n = 4$, and of $f \oplus g_1 \oplus g_4 \oplus \cdots \oplus g_{n-1}$, when $n \geqslant 5$, where f: $Z^3 \to Z_2$ is suitably defined. The proofs that the above functions do the job depend on a rather involved analysis of the subgroup of $(Z_2)^Z$ generated by $g_1, \cdots, g_n$. This analysis, together with additional details and results, is contained in [3].

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