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CLASSIFYING RELATIVE EQUILIBRIA. II

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We announce several theorems which suggest a minimal classification of relative equilibria in the planar n-body problem. These theorems also answer several questions on the nature of degenerate relative equilibria classes which were asked recently by S. Smale [3]. A summary of previous results can be found in an earlier paper [1]. It is a pleasure to thank S. Smale for encouragement in this work.

1. **Morse theory and relative equilibria.** We study the critical set of a real analytic function $\widetilde{V}_m < 0$ on a real analytic manifold X_m where $n \geqslant 3$ and $m = (m_1, \ldots, m_n) \in R^n_+$ are fixed. Critical points of $\widetilde{V}_m$ correspond in a 1-1 fashion to classes of relative equilibria. $\widetilde{V}_m$ always has a compact critical set which we may investigate by Morse theory even when degenerate critical points exist [2].

The integral singular homology of X_m (a manifold which is homeomorphic to a Stein manifold $P_{n-2}(C) - \widetilde{\Delta}_{n-2}$) is given by a recurrence relation [1]. This suggests that there is a uniform lower bound on the number of critical points of each index of $\widetilde{V}_m$ which is given by recurrence. As a first step toward classifying relative equilibria Theorem 1 gives such a relation.

In Theorem 2 we assert that $\widetilde{V}_m$ is a Morse function for any $n \geqslant 3$ and for almost all $m \in R^n_+$ (in the sense of Lebesgue measure).

Theorem 3 answers the question: Is $\widetilde{V}_m$ always a Morse function?

Finally, we examine the case of four masses to show how a degeneracy of $\widetilde{V}_m$ arises. An interpretation of Theorem 1 in the degenerate case sheds light on the creation and annihilation of relative equilibria.

2. **Main theorems.** In this paragraph for any i, $0 \leqslant i \leqslant 2n - 4$, let $\mu_i(n)$ denote a uniform lower bound to the number of critical points of $\widetilde{V}_m$

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with index equal to $2n - 4 - i$ whenever $\widetilde{V}_m$ is a Morse function. By [1, Theorem 2] the index of any critical point of $\widetilde{V}_m$ whether degenerate or not has $n - 2$ as a lower bound.

THEOREM 1. *For any $n \geqslant 3$ and for any i, $0 \leqslant i \leqslant n - 2$, $\mu_i(n) = (n - 1 - i)\,\mu_i(n - 1) + (2n - 2 - i)\,\mu_{i-1}(n - 1)$; and $\mu_i(n) = 0$ for $i > n - 2$.*

COROLLARY 1.1. *$\mu_i(n) = C_{n,i}(n - 1 - i)\,(n - 2)!$ for any i, $0 \leqslant i \leqslant n - 2$, and for any $n \geqslant 3$.*

Here $C_{n,i}$ is the binomial coefficient.

COROLLARY 1.2. *$\Sigma_{i=0}^{n-2}\,\mu_i(n) = [2^{n-1}(n - 2) + 1]\;(n - 2)!$ for any $n \geqslant 3$.*

Let $\beta_i = \operatorname{rank} H_i(P_{n-2}(C) - \widetilde{\Delta}_{n-2})$ for any i, $0 \leqslant i \leqslant 2n - 4$, and $n \geqslant 3$ where H_* is the integral singular homology. We write $A(t) > B(t)$ for any two polynomials $A(t)$, $B(t)$ provided that $A(t) - B(t) = (1 + t)\,C(t)$ where $C(t)$ has nonnegative coefficients. This relation subsumes the Morse inequalities.

COROLLARY 1.3. *$\Sigma_{i=0}^{n-2}\,\mu_i(n)t^i > \Sigma_{i=0}^{n-2}\,\beta_i t^i$ for any $n \geqslant 3$.*

Recently, S. Smale [3] has raised questions about the nature of the set of masses $\Sigma_n \subset R^n_+$ on which degeneracies of $\widetilde{V}_m$ arise. The next two theorems give some measure-theoretic properties of Σ_n.

THEOREM 2. *$\widetilde{V}_m$ is a Morse function for any $n \geqslant 3$ and for almost all masses $m \in R^n_+$ (in the sense of Lebesgue measure).*

COROLLARY 2.1. *There are only finitely many relative equilibria classes in the planar n-body problem for any $n \geqslant 3$ and for almost all masses $m \in R^n_+$.*

REMARK. It is an open question whether for some $n \geqslant 4$ and $m \in R^n_+$ there are infinitely many critical points of $\widetilde{V}_m$.

Theorem 2 shows that Σ_n has measure 0 for all $n \geqslant 3$. By [1, Theorem 4] we have $\Sigma_3 = \varnothing$. The next result shows that for $n \geqslant 4$ degeneracies arise.

THEOREM 3. *$\Sigma_n \neq \varnothing$ for any $n \geqslant 4$.*

3. **Classifying relative equilibria.** For any three positive masses there are precisely five critical points of $\widetilde{V}_m$ and these critical points are nondegenerate. Their distribution corresponds to that of the minimal classification given by Theorem 1.

For $n = 4$ masses a degeneracy arises in the following fashion. In the plane E^2 we place three unit masses at the vertices of an equilateral triangle with center of mass at the origin. We place at the origin an arbitrary fourth positive mass, m_4. It follows easily for all values of m_4 that this configuration is a relative equilibrium.

Let $m = (1, 1, 1, m_4)$ and let $x \in X_m$ be the relative equilibria class to which the above relative equilibrium belongs. Let $D^2\widetilde{V}_m(x)$, the hessian of $\widetilde{V}_m$ at x, a real symmetric bilinear form on T_xX_m, be considered a function of m_4. By direct calculation [2] we find that the hessian is degenerate if and only if m_4 equals the unique positive number m_4^* which is given by $m_4^* = (2 + 3\sqrt{3})/(18 - 5\sqrt{3}) < 1$.

For $m_4 < m_4^*$ the index of x (i.e. the index of the hessian of $\widetilde{V}_m$ at x) equals 4 and x is a nondegenerate local maximum of $\widetilde{V}_m$. For $m_4 \geqslant m_4^*$ the index of x equals 2. When $m_4 = m_4^*$ the dimension of the nullspace of the hessian equals 2. This is the maximum degeneracy possible for four masses.

These considerations suggest the following interpretation of Theorem 1 whenever $\widetilde{V}_m$ has isolated degenerate critical points.

For any $n \geqslant 4$ let $m \in R_+^n$ be such that $\widetilde{V}_m$ has only *isolated* critical points. Let $c_1 < \ldots < c_r < 0$ be the critical values of $\widetilde{V}_m$. Set $c_0 = -\infty$ and for any j, $1 \leqslant j \leqslant r$, define $W_j = \widetilde{V}_m^{-1}(c_{j-1}, c_j)$. Let Λ_j be the set of critical points at level j, $1 \leqslant j \leqslant r$. Finally, for any i, $0 \leqslant i \leqslant 2n - 4$, define $\tau_i(n, m)$ by

$$\tau_i(n, m) = \sum_{j=1}^{r} \operatorname{rank} H_{2n-4-i}(W_j \cup \Lambda_j, W_j).$$

By [1, Theorem 2] we have $\tau_i(n, m) = 0$ for any $i > n - 2$.

THEOREM 4. *For any $n \geqslant 4$ and any $m \in R_+^n$ for which $\widetilde{V}_m$ has only isolated critical points, $\tau_i(n, m) \geqslant \mu_i(n)$ for any i, $0 \leqslant i \leqslant 2n - 4$.*

COROLLARY 4.1. $\Sigma_{i=0}^{n-2} \tau_i(n, m)\, t^i > \Sigma_{i=0}^{n-2} \mu_i(n)\, t^i$ *for any $n \geqslant 4$.*

REFERENCES

1. J. I. Palmore, *Classifying relative equilibria.* I, Bull. Amer. Math. Soc. 79 (1973), 904–908. MR 47 #9922.

2. ———, *Relative equilibria of the n-body problem*, Thesis, University of California, Berkeley, Calif., 1973.

3. S. Smale, *Problems in dynamical systems and celestial mechanics*, (preprint). University of California, Berkeley, Calif., 1974.

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