BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, March 1975

A 4-MANIFOLD WHICH ADMITS NO SPINE

BY YUKIO MATSUMOTO[1]

Communicated by Edgar Brown, Jr., October 26, 1974

1. This note is to present a new example which reveals the impossibility of embedding a 2-torus in a 4-manifold.

THEOREM 1. *There exists a compact 4-dimensional PL manifold* W^4 *with boundary satisfying the following conditions*: (i) W^4 *is homotopically equivalent to the 2-torus* $T^2 = S^1 \times S^1$, *and* (ii) *no homotopy equivalence* $T^2 \longrightarrow W^4$ *is homotopic to a PL embedding.*

By a PL embedding is meant one which is not necessarily locally flat.

Theorem 1 is an application of the codimension two surgery theory developed in our previous papers [4], [5], [6]. The phenomena of "total spinelessness" in higher dimensions (with finite π_1's) were found by Cappell and Shaneson [2] using another method of surgery[2] [1].

A calculation in our proof leads to another consequence concerned with submanifolds in codimension two. Let K^{4n} denote a product $CP_2 \times \cdots \times CP_2$ of n-copies of the complex projective plane CP_2.

THEOREM 2. *For each* $n \geqslant 0$, *there exists a locally flat embedding* $h_{(4n)}$ *of* $K^{4n} \times S^1$ *into the interior of* $K^{4n} \times D^2 \times S^1$, *which is homotopic to the zero cross section* $K^{4n} \times \{0\} \times S^1$, *but is not locally flatly concordant to a splitted embedding.*

A *splitted embedding* (with respect to a point $*$ of S^1) means a locally flat embedding $f: K^{4n} \times S^1 \longrightarrow K^{4n} \times D^2 \times S^1$ such that (i) f is transverse regular to $K^{4n} \times D^2 \times \{*\}$ so that the intersection $M^{4n} = f(K^{4n} \times S^1) \cap K^{4n} \times D^2 \times \{*\}$ is a closed manifold, and (ii) the inclusion $M^{4n} \longrightarrow K^{4n} \times D^2 \times \{*\}$ is a homotopy equivalence.

Theorem 2 contrasts with Farrell and Hsiang's result [3] which may be

---

AMS (MOS) subject classifications (1970). Primary 55C45, 57C65, 18F25; Secondary 55A25, 55C35.

[1]The author is partially supported by the Fūjukai Foundation.

[2]Their theory (with Γ-groups) and ours (with P-groups) are not the same but both admit a more general unifying algebraic treatment [7].

considered as the splitting theorem in codimension $\geqslant 3$.

2. **Construction of W^4.** Let $h\colon S^1 \longrightarrow S^1 \times D^2$ be an embedding indicated in Figure 1. Essentially the same embedding $S^1 \longrightarrow S^1 \times S^2$ was used by Mazur [8] to construct a contractible 4-manifold.

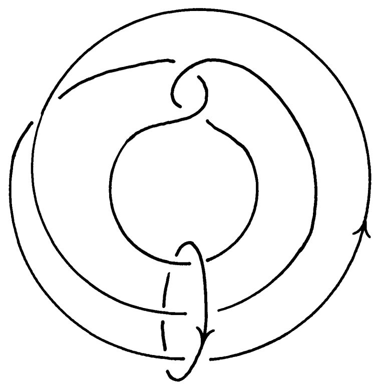

FIGURE 1. Mazur's embedding

Extend h to a framed embedding $\bar{h}\colon S^1 \times D^2 \longrightarrow S^1 \times D^2$ in such a way that $\bar{h}$ followed by the natural inclusion $S^1 \times D^2 \longrightarrow S^3$ is isotopic to a trivial knot with a trivial framing. Our manifold W^4 is the mapping torus of the framed embedding $\bar{h}$. More precisely, W^4 is obtained from a product $S^1 \times D^2 \times [0, 1]$ by identifying $(x, \xi) \times \{1\}$ with $\bar{h}(x, \xi) \times \{0\}$ for each $(x, \xi) \in S^1 \times D^2$. Since h is homotopic to the zero cross section $S^1 \times \{0\} \longrightarrow S^1 \times D^2$, W^4 is homotopically equivalent to T^2.

Moreover, the embedding $h_{(4n)}$ in Theorem 2 is nothing other than $\mathrm{id}_K \times h\colon K^{4n} \times S^1 \longrightarrow K^{4n} \times S^1 \times D^2$, h being Mazur's one.

3. **Sketch of proof.** We first give some generalities. Suppose a compact connected oriented PL $2n+2$-manifold V^{2n+2} has the same simple homotopy type as an oriented Poincaré complex of formal dimension $2n \geqslant 6$. Let $\pi \longrightarrow \pi'$ denote the *associated* (*onto*) *homomorphism* with V^{2n+2} defined to be $\pi_1(V - L) \longrightarrow \pi_1(V)$, where L^{2n} is an exterior n-connected (i.e. taut) $2n$-submanifold of V^{2n+2} [4]. The kernel of $\pi \longrightarrow \pi'$ is generated by a (specified) central element t represented by a fiber of the associated S^1-bundle with a 2-disk bundle neighbourhood N of L^{2n}.

A $(-1)^n$-*Seifert form* over $\pi \longrightarrow \pi'$ is, by definition, a (not necessarily nonsingular) $(-1)^n t$-Hermitian form defined over $\mathbf{Z}\pi$ which becomes nonsingular over $\mathbf{Z}\pi'$ (after tensored with $\mathbf{Z}\pi'$).

Then the left $Z\pi$-module $\pi_{n+1}(V - L, N - L)$ is proved to carry a $(-1)^n$-Seifert form whose class in $P_{2n}(\pi \longrightarrow \pi')$[3], the "Witt group" of $(-1)^n$-Seifert forms over $\pi \longrightarrow \pi'$, does not depend on L. Denote the class by $\eta(V) \in P_{2n}(\pi \longrightarrow \pi')$. Then $\eta(V) = 0$ if and only if V admits a locally flat spine [6].

Now with the notations of §2, the product $W^4 \times CP_2$ has the homotopy type of $T^2 \times CP_2$. The associated homomorphism with it is $\{Z \times Z \times Z \longrightarrow Z \times Z\} = (Z \longrightarrow 1) \times Z \times Z$, and the obstruction element $\eta(W^4 \times CP_2)$ is proven to be in the image of the injective homomorphism

$$j_*: P_6((Z \longrightarrow 1) \times Z) \longrightarrow P_6((Z \longrightarrow 1) \times Z \times Z).$$

Let $\eta' = j_*^{-1}(\eta(W^4 \times CP_2))$.

Lemma 1. *The element η' of $P_6((Z \longrightarrow 1) \times Z)$ is represented by a (-1)-Seifert form (G, λ, μ) given by: $G = \Lambda x_1 \oplus \Lambda x_2$, $\lambda(x_1, x_2) = -s^{-1}$, $\mu(x_1) = s - 1$, $\mu(x_2) = -1$, where $\Lambda = Z[t, t^{-1}, s, s^{-1}]$, t (or s) denoting the positive generator of the first (or the second) Z of $(Z \longrightarrow 1) \times Z$.*

Remark. The matrix $(\lambda(x_i, x_j))$ of the (-1)-Seifert form of Lemma 1 is

$$\begin{pmatrix} (s-1) - (s^{-1} - 1)t, & -s^{-1} \\ st, & -1 + t \end{pmatrix},$$

the determinant of which coincides (up to units) with the Alexander polynomial of Mazur's link (Figure 1) calculated by the method of Torres and Fox [9].

Lemma 2. *η' is not in the image of*

$$i_*: P_6(Z \longrightarrow 1) \longrightarrow P_6((Z \longrightarrow 1) \times Z).$$

The proof of Theorem 1 goes as follows. Suppose that there were a spine $T_0^2 \subset W^4$. T_0^2 may be assumed to be locally flat except at one point. The product $T_0^2 \times CP_2$ is a spine of $W^4 \times CP_2$ whose singularity is of the type (knot cone) $\times\, CP_2$. Since $\pi_1(\{pt\} \times CP_2) = \{1\}$, this singularity is replaced by a knot cone singularity over a knotted 5-sphere in a 7-sphere [4], [6, §6.4]. This implies that the $\eta(W^4 \times CP_2)$ is in the image of $j_* \circ i_*$, since $P_6(Z \longrightarrow 1)$ is isomorphic to the (7, 5)-knot cobordism group [6]. However, this contradicts Lemma 2.

[3]This notation slightly differs from the original one [6].

REMARK. If we start the construction with the embedding indicated in Figure 2, we will obtain $W^{4'}$ which *admits* a locally flat spine.

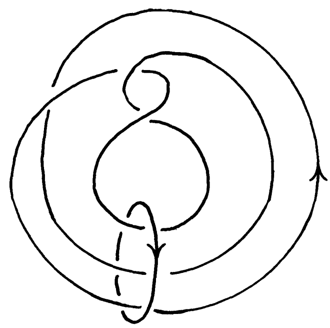

FIGURE 2. False embedding

REFERENCES

1. S. E. Cappell and J. L. Shaneson, *The codimension two placement problem and homology equivalent manifolds*, Ann. of Math. (2) **99** (1974), 277–348.

2. ———, *Totally spineless manifolds*, Topology (to appear).

3. F. T. Farrell and W. C. Hsiang, *A geometric interpretation of the Künneth formula for algebraic K-theory*, Bull. Amer. Math. Soc. **74** (1968), 548–553. MR **37** #274.

4. M. Kato and Y. Matsumoto, *Simply connected surgery of submanifolds in codimension two*. I, J. Math. Soc. Japan **24** (1972), 586–608. MR **46** #6369.

5. Y. Matsumoto, *Surgery and singularities in codimension two*, Proc. Japan Acad. **47** (1971), 153–156. MR **45** #6008.

6. ———, *Knot cobordism groups and surgery in codimension two*, J. Fac. Sci. Univ. Tokyo, Sect. IA **20** (1973), 253–317.

7. ———, *Some relative notions in the theory of Hermitian forms*, Proc. Japan Acad. **49** (1973), 583–587.

8. B. Mazur, *A note on some contractible* 4*-manifolds*, Ann. of Math. (2) **73** (1961), 221–228. MR **23** #A2873.

9. G. Torres and R. H. Fox, *Dual presentation of the group of a knot*, Ann. of Math. (2) **59** (1954), 211–218. MR **15**, 979.

DEPARTMENT OF MATHEMATICS, COLLEGE OF GENERAL EDUCATION, UNIVERSITY OF TOKYO, KOMABA, MEGURO-KU, TOKYO, 153, JAPAN