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AN ENTROPY EQUIDISTRIBUTION PROPERTY FOR A MEASURABLE PARTITION UNDER THE ACTION OF AN AMENABLE GROUP

BY J. C. KIEFFER

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Throughout this note let G be an arbitrary discrete amenable group. Let $(\Omega, \mathcal{M}, \lambda)$ be a probability space. Let $\mathcal{A}$ be the automorphism group of $(\Omega, \mathcal{M}, \lambda)$. Let T: $G \longrightarrow \mathcal{A}$ be a group homomorphism. We call T an action of G on Ω. For each $g \in G$, let T^g be the image of g in $\mathcal{A}$ under T. Then T^g is a measurable, measure-preserving, invertible map from Ω to itself.

If Q is a partition of Ω and $\omega \in \Omega$, let $Q(\omega)$ be the element of Q which contains ω. If E is a set let $|E|$ denote the cardinality of E.

Let K be a subgroup of G. A net $\{A_\alpha\}$ of finite nonempty subsets of K is said to satisfy property P with respect to K if $\lim_\alpha |A_\alpha|^{-1} |gA_\alpha \cap A_\alpha| = 1, g \in K$. (Since K is amenable, such a net $\{A_\alpha\}$ exists; see [3].)

Let P be a measurable partition of Ω with finite entropy. If E is a finite nonempty subset of G, let $h_P(E) \in L^1(\Omega)$ be defined as follows:

$$h_P(E)(\omega) = -\log \lambda \left[\left\{ \bigvee_{g \in E} (T^g)^{-1} P \right\} (\omega) \right], \qquad \omega \in \Omega.$$

The following generalization of the Shannon-McMillan theorem may be found in [4] and [8]: Let $G = Z^k$, where Z is the group of integers and k is a positive integer. For $n = 1, 2, \cdots$, let $A_n = \{(x_1, x_2, \cdots, x_k) \in Z^k: 0 \leqslant x_i \leqslant n, i = 1, 2, \cdots, k\}$. Then $\{|A_n|^{-1} h_P(A_n)\}$ converges in $L^1(\Omega)$ as $n \longrightarrow \infty$.

In [7] it is shown that if G is the group of dyadic rationals modulo one, and if A_n is the cyclic subgroup of G generated by 2^{-n}, then $\{|A_n|^{-1} h_P(A_n)\}$ converges in $L^1(\Omega)$ as $n \longrightarrow \infty$. The authors of [7] conjectured that this property generalizes to a general countable abelian group.

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It is the purpose of this note to announce the following theorem which generalizes these results, and settles the above conjecture. (The proofs of Theorems 1–4 will appear elsewhere.) Following [7], we call Theorem 1 the entropy equidistribution property of a measurable partition under the action of an amenable group.

THEOREM 1. *Let K be a subgroup of the amenable group G. There exists a K-invariant function $h(P, T, K) \in L^1(\Omega)$ such that for every net $\{A_\alpha\}$ satisfying property $\mathcal{P}$ with respect to K, $\lim_\alpha |A_\alpha|^{-1} h_P(A_\alpha) = h(P, T, K)$ in $L^1(\Omega)$.*

The main tool used in proving Theorem 1 is the following generalized ergodic theorem which appears in [1]: If K is a subgroup of G, $\{A_\alpha\}$ is a net satisfying property $\mathcal{P}$ with respect to K, and $f \in L^1(\Omega)$, then $\{|A_\alpha|^{-1} \Sigma_{g \in A_\alpha} f \cdot T^g\}$ has a limit in $L^1(\Omega)$ which is K-invariant.

Define $\mathcal{H}(P, T, K) = \int h(P, T, K)\, d\lambda$. Define $\mathcal{C}(K) = \{M \in \mathcal{M}: \lambda[T^g(M) \,\Delta\, M] = 0, g \in K\}$.

THEOREM 2. *If K_1 and K_2 are subgroups of G such that $K_1 \subset K_2$, then $\mathcal{H}(K_2) \leqslant \mathcal{H}(K_1)$.* Equality holds if and only if $E[h(P, T, K_1)|\mathcal{C}(K_2)] = h(P, T, K_2)$.

THEOREM 3. *If K is a subgroup of G, there exists a countable subgroup L of K such that if L' is any subgroup satisfying $L \subset L' \subset K$, then $h(P, T, L') = h(P, T, K)$.*

THEOREM 4. *Let K be a subgroup of G. Let $\mathcal{K}$ be a family of subgroups of K which is directed by inclusion ($\supset$), and whose union is K. Then $\lim_{L \in \mathcal{K}} h(P, T, L) = h(P, T, K)$ in $L^1(\Omega)$, and $\mathcal{H}(P, T, K) = \inf_{L \in \mathcal{K}} \mathcal{H}(P, T, L)$.*

As an application of the foregoing results, we can define the entropy $\mathcal{H}(T)$ of the action T of the amenable group G on Ω as follows: $\mathcal{H}(T) = \sup_P \mathcal{H}(P, T, G)$, where the supremum is over all measurable partitions P of Ω with finite entropy. This definition extends that given in [2] for $G = Z^k$. The entropy as we have defined it is an invariant under isomorphism. Conversely, it may be possible to generalize Ornstein's results [6] and show that generalized Bernoulli schemes (see [5] for definition) with the same entropy are isomorphic. The entropy equidistribution property (Theorem 1 above) might serve as a basic tool for proving this.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-ROLLA, ROLLA, MISSOURI 65401