BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, March 1975

SEMICONTINUITY OF KODAIRA DIMENSION

BY D. LIEBERMAN[1] AND E. SERENESI

Communicated by Edgar Brown, Jr., January 10, 1975

Let X be a compact analytic space (or a complete algebraic variety) and let L be a line bundle on X and denote by $f_i\colon X \longrightarrow \mathbf{P}^N$ the rational map defined by the global sections of $L^{\otimes i}$. The L-dimension of X, $K(X, L)$ is defined by

$$K(X, L) = \overline{\lim_{i\to\infty}} (\dim(f_i(X))$$

with the convention $K(X, L) = -\infty$ if $L^{\otimes i}$ has no nontrivial sections for all $i > 0$. In the particular case when X is nonsingular and $L = \Omega$ is the canonical bundle, the invariant $K(X) = K(X, \Omega)$ is called the canonical (or Kodaira) dimension of X and is the fundamental invariant in the classification of surfaces. Recent works by Ueno [4] and Iitaka [1], [2] have studied $K(X, L)$ for higher dimensional varieties. A fundamental open question is the behavior of $K(X, L)$ under deformations of (X, L). When X is a smooth surface the plurigenera (and hence the Kodaira dimension) are deformation invariant [1], and Iitaka has constructed a family of threefolds X_t with $K(X_0) = 0$ and $K(X_t) = -\infty$, $t \neq 0$.

Our main result is

THEOREM. *Given X_0 a compact analytic space (or complete algebraic variety) and L_0 a line bundle on X_0 satisfying*

(1) *$L_0^{\otimes i}$ is spanned by its global sections for some $i > 0$,*

(2) *$K(X_0, L_0) = \dim(X_0)$,*

and (X_t, L_t) is any (flat) deformation of (X_0, L_0), then $K(X_t, L_t) = K(X_0, L_0)$.

When X_0 is a smooth surface and $L_0 = \Omega_0$ it was shown by Mumford [3] that hypothesis (1) on L_0 is implied by (2). For general L_0 hypothesis

---

AMS (MOS) subject classifications (1970). Primary 14D15, 32G05.

Key words and phrases. Kodaira dimension, semicontinuity, deformation.

[1]Sloan Foundation Fellow, partially supported by NSF GP-28323A3.

Copyright © 1975, American Mathematical Society

(1) is not implied by (2); however when X_0 is smooth and $L_0 = \Omega_0$ the first hypothesis might be unnecessary.

The general line of argument is the following:

Given $\pi: X \longrightarrow S$ proper and flat and $L \longrightarrow X$ a line bundle one defines a relative L-dimension $K(X|S, L)$ as follows. Let $f_i: X \longrightarrow \mathrm{Proj}(\pi_* L^{\otimes i})$ be the rational map defined by $\pi^*(\pi_* L^{\otimes i}) \longrightarrow L^{\otimes i}$. Let

$$K(X|S, L) = \overline{\lim}(\dim f_i(X)) - \dim(S)$$

(or $-\infty$ if $\pi_*(L^{\otimes i}) = 0$ for all $i > 0$).

PROPOSITION 1. *$K(X_s, L_s) \geqslant K(X|S, L)$ for all $s \in S$ with equality for $s \in W$ a nonempty c-open subset of S (i.e. W is the complement of a countable union of subvarieties).*

As an immediate corollary one sees that the L-dimension is upper semicontinuous in the topology defined by c-open sets. The set W and its complement may both be dense, e.g. taking $X_s \xrightarrow{\sim} X_0$ a curve of genus $g > 0$, S the Jacobian of X_0 and L_s the canonical family of degree zero bundles, one finds $W = S -$ {points of finite order}.

The main theorem follows from

PROPOSITION 2. *If $K(X_s, L_s) = \dim(X_s)$ and $L_s^{\otimes i}$ is spanned by its global sections for some $i > 0$ then $s \in W$.*

More generally if $d = K(X_s, L_s) \leqslant \dim(X_s)$ and for some $i > 0$ the map $f_{i,s}: X_s \longrightarrow \mathbf{P}^N$ given by $L_s^{\otimes i}$ is everywhere defined, $\dim(f_{i,s}(X_s)) = d$, and $\dim(\mathrm{supp}\, R^1 f_{is*}(\mathcal{O})) < d$ then $s \in W$. Thus for pairs (X_s, L_s) satisfying the preceding hypotheses, the L-dimension can only go up under deformation.

REFERENCES

1. S. Iitaka, *Deformations of compact complex surfaces*. II, J. Math. Soc. Japan 22 (1970), 247–261. MR 41 #6252.

2. ———, *On D-dimensions of algebraic varieties*, J. Math. Soc. Japan 23 (1971), 356–373. MR 44 #2749.

3. D. Mumford, *The canonical ring of an algebraic surface*, Ann. of Math. (2) 76 (1962), 612–615.

4. K. Ueno, *On Kodaira dimensions of certain algebraic varieties*, Proc. Japan Acad. 47 (1971), 157–159. MR 45 #1911.

DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, WALTHAM, MASSACHUSETTS 02154