BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, March 1975

AUTOMORPHISMS OF SOLVABLE GROUPS

BY S. BACHMUTH AND H. Y. MOCHIZUKI[1]

Communicated by Hyman Bass, October 15, 1974

We state an analogue of Tits' theorem on linear groups [3] as

CONJECTURE. Let G be a finitely generated (f.g.) solvable group. Then, any f.g. subgroup of the automorphism group of G is solvable by finite or contains a noncyclic free group.

As preliminary evidence, it was noticed by G. Baumslag and the authors that the Conjecture is correct when G is nilpotent-by-abelian.

ZG denotes the integral group ring of a group G and $Q(D)$ the division ring of quotients of an Ore domain D. If ZG is an Ore domain and U a group of matrices over $Q(ZG)$, we say U has a (right) common denominator if there is $b \in ZG$ such that each entry in a matrix of U has the form ab^{-1}, $a \in ZG$.

Henceforth these notations hold. F a free group whose rank will be specified, $R \neq \{1\}$ a normal subgroup of F, $\gamma_n(R)$ the nth term of the lower central series of R, $R' = \gamma_2(R)$, $H = F/R$, $G = F/R'$, $A(G)$ the automorphism group of a group G, $A(G_1; G_2)$ the kernel of $A(G_1) \longrightarrow A(G_2)$.

Theorem 1 is joint work with E. Formanek.

THEOREM 1. *Let F have rank two and assume $Z(F/R)$ is a domain with $R \leqslant F'$. Then $A(F/R'; F/F')$ consists entirely of inner automorphisms.*

COROLLARY. *Let F/R be as in Theorem 1 and also assume F/R is solvable. Then $F/\gamma_n(R)$ satisfies the Conjecture.*

Problem 1. Let F have rank two and $H = F/R$ be a solvable group such that ZH is an Ore domain, and let U be a group of units of $Q(ZH)$ which has a common denominator. Is U a solvable group?

An affirmative answer to Problem 1 would yield a proof of the Corollary independent of Theorem 1. It seems reasonable to suspect that a group U of units in $Q(ZH)$, ZH an Ore domain, having a common denominator is in fact a conjugate of a group of units of ZH.

AMS (MOS) subject classifications (1970). Primary 20F55, 20E99; Secondary 20H25.

[1] Both authors were supported by NSF Grant No. GP-38968.

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THEOREM 2. *Let H be a poly-(infinite cyclic) group, and let U be a f.g. subgroup of* $SL_2(Q(ZH))$ *which has a common denominator. Then, U contains a noncyclic free subgroup, or U is a finite extension of a radical group.*

A radical group has an ascending series terminating in the group such that each factor group of the series is locally nilpotent. As a corollary of Theorem 2, we have

THEOREM 3. *Let F have rank three and $H = F/R$ be poly-(infinite cyclic). Let A be a f.g. subgroup of $A(G; H)$.* (i) *If A satisfies the maximum condition on abelian subgroups, then A contains a noncyclic free subgroup or A is polycyclic by finite.* (ii) *If A satisfies the minimum condition on abelian subgroups, then A is a Černikov group (i.e., abelian by finite satisfying the minimum condition on subgroups).* (iii) *If each normal subgroup of A is f.g., then A contains a noncyclic free subgroup or A is polycyclic by finite. Moreover,* (i)–(iii) *are true if A is a f.g. subgroup of $A(F/\gamma_n(R))$.*

Problem 2. Can one improve Theorem 3 to conclude that A contains a noncyclic free group or is solvable-by-finite without restrictions on A?

A positive answer to Problem 2 would follow if one could show that a locally solvable subgroup of $SL_2(Q(ZH))$ is solvable; e.g., by showing that there is a bound on the derived length of a solvable subgroup of $SL_2(Q(ZH))$. There is one nice case in which we can answer Problem 2.

THEOREM 4. *Let F have rank three and let F/R be torsion-free nilpotent of class two. Then, $F/\gamma_n(R)$ satisfies the Conjecture.*

Problem 3. Let G be a solvable group such that ZG is an Ore domain. If U is a f.g. subgroup of $SL_2(Q(ZG))$ which has a common denominator, what conclusion can one draw concerning U?

THEOREM 5. *Let F have rank n, and $H = F/R$ be free metabelian. Then, any f.g. subgroup of* $SL_2(ZH)$ *is solvable-by-finite or contains a noncyclic free subgroup.*

Our concluding result adds to the evidence for the Conjecture by indicating the prevalence of free subgroups.

THEOREM 6. *Let F have rank three and $H = F/R$ be any group for which $Z(H)$ has no nonzero zero divisors. Let A_i $(1 \leqslant i \leqslant 3)$ be the (abelian)* *subgroup of $A(G; H)$ whose nonidentity elements leave all but the ith generator fixed. Then, the subgroup of $A(G; H)$ generated by the A_i is the free product of the A_i.*

The restriction on the rank of F is not essential. A similar more complicated result holds for F of rank > 3.

The pioneering work of Magnus described the relevant automorphism groups as automorphisms of free modules in an explicit way. The work announced here is a start in exploiting Magnus' representations in noncommutative contexts. In our proofs we relied on the structure of a skew-polynomial domain $K[x]$, K a division ring, and its division ring of quotients $Q(K[x])$ with the induced discrete valuation. The decomposition of $SL_2(Q(K[x]))$ and $SL_2(K[x])$ as amalgamated free products of groups due to Ihara and Nagao, respectively [2], were employed, and also the subgroup theorems for amalgamated products of Karrass and Solitar [1].

We conclude with a more specific conjecture which has a deeper relationship with Tits' theorem. Henceforth a linear group will mean a group of matrices over a commutative Noetherian ring. We call a group poly-L if the group has a finite subnormal series such that the factors are either linear or abelian groups. Our suggestion is that

"The group of automorphisms of a f.g. solvable group is poly-L."

F.g. nilpotent-by-abelian groups have automorphism groups which are poly-L (this includes all polycyclic groups). Theorem 1 tells us that the automorphism groups of a large class of 2-generator nilpotent-by-torsion-free solvable groups are poly-L.

ACKNOWLEDGEMENT. We thank G. Baumslag for his interest.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106