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GROUPS OF DIFFEOMORPHISMS OF R^n AND THE FLOW OF A PERFECT FLUID

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Introduction. In this paper it is shown how groups of diffeomorphisms can be used to give a new proof of the short-time existence and uniqueness of solutions to the Euler equations for a perfect fluid over R^n:

$$\text{(E)} \qquad \partial U_t/\partial t + DU_t \cdot U_t = -\text{grad}\, P_t, \qquad \text{div}\, U_t = 0.$$

T. Kato [4] and H. Swann [6] achieved similar results by showing that solutions to the Navier-Stokes equations for viscous flows converge to solutions of E in $H^s = L^2_s$ as the viscosity approaches zero. Our results differ from theirs in that the solutions we establish possess a wider variety of asymptotic conditions at infinity.

The proofs of the theorems shall appear elsewhere [1], [2].

Future work includes the extension of these results to flows over noncompact Riemannian manifolds and the study of the related problem of exterior flows in R^n.

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1. **Groups of diffeomorphisms.** We use the standard multi-index notation for differential operators. $\sigma(x) = (|x|^2 + 1)^{1/2}$ and JF denotes the Jacobian determinant.

DEFINITION 1.1. Let $|\ |_p$ be the standard L^p norm on R^n. Then define

$$\|f\|_{o,p,\delta} \doteq |\sigma^\delta \cdot f|_p$$

and

$$\|f\|_{s,p,\delta} = \sum_{|\alpha| \leq s} \|D^\alpha f\|_{o,p,\delta + |\alpha|}.$$

Also, define $M^p_{s,\delta}(R^n, R^m)$ to be the completion of $C^\infty_0(R^n, R^m)$ with re-

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spect to $\| \|_{s,p,\delta}$. These spaces are studied in some detail in §2. In particular it is shown that for some choices of p, s, and δ these spaces satisfy the Sobolev embedding theorem, have the Schauder ring property, and behave regularly with respect to the Laplacian.

DEFINITION 1.2. Let $h: R^n \longrightarrow R^m$ be smooth (not necessarily bounded).

$$M^p_{s,\delta}(h) = \{f: R^n \longrightarrow R^m \,|\, (f - h) \in M^p_{s,\delta}(R^n, R^m)\}.$$

$$C^k(h) = \{f: R^n \longrightarrow R^m \,|\, \|g - h\|_{C^k} < \infty\}.$$

We denote the identity map on R^n by I.

A fundamental composition theorem in this context is

THEOREM 1.3. *Let* $p > 1, s > n/p + 1$, *and* $U(I) = \{g \in M^p_{s,\delta}(I) |$ $\inf_{R^n} Jg(x) > 0\}$. *Then if* $f: R^n \longrightarrow R^m$ *is such that for each* k, $\sigma^{|k|+\gamma-1}D^k f$ *is a bounded map, composition is continuous.*

$$M^p_{k,\gamma}(f) \oplus U(1) \longrightarrow M^p_{k,\gamma}(f), \qquad (f, g) \longrightarrow f \circ g.$$

DEFINITION 1.4. $D^p_{s,\delta} = \{f \in M^p_{s,\delta}(I) | f^{-1} \in M^p_{s,\delta}(I)\}$.

THEOREM 1.5. *Let* $p > 1, \delta \geqslant 0$, *and* $s > n/p + 1$. *Then*

(1) $D^p_{s,\delta} \subset M^p_{s,\delta}$ *is an open topological group.*

(2) *Composition is a jointly continuous map*:

$$M^p_{s,\delta} \oplus D^p_{s,\delta} \longrightarrow M^p_{s,\delta}, \qquad (f, g) \longrightarrow f \circ g.$$

REMARK. Right composition is a linear continuous map and hence smooth. However, using arguments of Kato [5] one can show left composition is not Hölder continuous for any $\alpha > 0$. Thus the above result is sharp.

Adopting the terminology of manifolds of maps, we define the tangent bundle of $D^p_{s,\delta}$ as follows.

DEFINITION 1.6. Let $z \in D^p_{s,\delta}$. Then $T_z D_{s,\delta} = \{f: R^n \longrightarrow (R^n) |$ $f = (z, q)$ where $q \circ z^{-1} \in M^p_{s,\delta}(R^n, R^n)\}$ and $TD^p_{s,\delta} = UT_z D^p_{s,\delta}$ with the usual differential structure of a tangent bundle. It can be continuously identified with $D^p_{s,\delta} \times M^p_{s,\delta}$ since $Rz^{-1}: (z, q) \longrightarrow (I, q \circ z^{-1})$ maps $T_z D^p_{s,\delta} \longrightarrow T_I D^p_{s,\delta} \simeq M^p_{s,\delta}$ according to Theorem 1.5. Note that this map is not smooth.

DEFINITION 1.7. $F^p_{s,\delta} = \{f \in D^p_{s,\delta}: Jf = 1\}$ where Jf is the Jacobian determinant of f.

THEOREM 1.8. *Let* $p > n/(n-2)$, $s > n/p + 1$ *and* $0 \leqslant \rho \leqslant -2 + n(p-1)/p$. *Then if* $\delta = 1 + \rho$, $F^p_{s,\delta}$ *is a smooth submanifold of* $D^p_{s,\delta}$. *Also if* $z \in F^p_{s,\delta}$, $V \in T_z F^p_{s,\delta}$ *if* $V \in T_z D^p_{s,\delta}$ *and* $\operatorname{div}(V \circ z^{-1}) = 0$.

2. **The flow of a perfect fluid.** Using the technique of [3], the geodesic spray on R^n can be lifted to a smooth spray on $M^p_{s,\delta}(I)$. The corresponding geodesics in $M^p_{s,\delta}(I)$ are straight lines in this affine Banach space. Now $F^p_{s,\delta}$ will inherit a geodesic spray via the projection of $TM^p_{s,\delta}|_{F^p_{s,\delta}}$ onto $TF^p_{s,\delta}$. This spray is smooth if the projection is smooth as a bundle map. Euler flows on R^n correspond to geodesics on $F^p_{s,\delta}$ with respect to this spray. Thus, since short-time existence and uniqueness of geodesics of a smooth spray are guaranteed, the short-time existence and uniqueness of solutions to the Euler equations follow from showing the projection is smooth.

DEFINITION 2.1. Let $TD^p_{s,\delta}$ and $TF^p_{s,\delta}$ be as defined in the last section, and define $P: TD^p_{s,\delta}|_{F^p_{s,\delta}} \longrightarrow TF^p_{s,\delta}$ by

$$P(z, x) = (z, x - (\operatorname{grad} \Delta^{-1} \operatorname{div} X \circ z^{-1}) \circ z).$$

THEOREM 2.2. *P is a smooth map.*

Thus we can summarize our results on solutions to the Euler equations as follows.

THEOREM 2.3. *Let* $n > 2$, $p > n/(n-2)$, $1 \leqslant \delta \leqslant -1 + n((p-1)/p)$, *and* $s > n/p + 1$, *then for* $U_0 \in M^p_{s,\delta}$ *with* $\operatorname{div}(U_0) = 0$, *there is a unique short-time solution* U_t *to the Euler equations in* $M^p_{s,\delta}$ *starting with* U_0. *Also these solutions depend continuously on initial conditions.*

REMARK. Not all of the solutions guaranteed by the above theorem are physical in the sense that they have bounded energy. However, as is shown in (1), such solutions can be shown to exist for any $p > 1$ and $s > n/p + 1$ by choosing δ appropriately.

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