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ALGEBRAS OF ANALYTIC FUNCTIONS ON DEGENERATING RIEMANN SURFACES

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I. **Introduction.** By a Riemann surface we mean a finite bordered Riemann surface. For a Riemann surface S denoted by $A(S)$ the supremum normed Banach algebra of functions continuous on S and analytic on the interior of S. For any two Banach spaces A and B define $d(A, B) = \log \inf \{\|T\| \, \|T^{-1}\|;\ T$ a continuous invertible linear map of A onto $B\}$. For S_1 and S_2 homeomorphic Riemann surfaces define $d(S_1, S_2) = d(A(S_1), A(S_2))$. It is known [7] that d defines a metric on $R(S_1)$, the Riemann space of S_1, the space of conformal equivalence classes of Riemann surfaces homeomorphic to S_1, and that the topology induced by this metric is the same as that induced by the Teichmüller metric. The metric space $(R(S_1), d)$ is not complete. In this note we present properties of the ideal elements that are introduced in forming the completion of the metric space $(R(S_1), d)$. Proofs of these and related results will appear in a later publication.

The main result is that the new elements are connected degenerate Riemann surfaces. In fact, the results presented strongly suggest (but do not prove) that the completion of $(R(S_1), d)$ is formed by adjoining to $R(S_1)$ exactly those elements obtained by "pinching to a point" of closed noncontractible curves on surfaces in $R(S_1)$.

On an informal geometric level these results are related to results on degeneration of compact surfaces [4] and results on boundary points of Teichmüller space [1], [2], [5].

II. **An example.** The following example illustrates many of the phenomena described in Theorem 2. For $0 < r < 1$ let $S_r = \{r \leqslant |z| \leqslant 1\}$. Let $A_r = A(S_r)$. Let S_0 be two closed disks with their centers identified, and let A_0 be the algebra of continuous functions on S_0 which are "analytic" on the interior.

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THEOREM 1. *The set* $\{A_r\}_{0\leq r<1}$ *is a complete metric space with respect to the metric* d. *This space is homeomorphic (in the natural way) to* $[0, 1)$.

This example contains a great deal of negative information about the metric d.

COROLLARY. *The metric space* $(R(S_{1/2}), d)$ *is not complete. Hence the metric* d *on* $R(S_{1/2})$ *is not equivalent to the Teichmüller metric.*

COROLLARY. *Given* ϵ *positive one can find Banach algebras* A *and* B *with maximal ideal spaces* $M(A)$ *and* $M(B)$ *such that* $d(A, B) < \epsilon$, *but the homology groups of* $M(A)$ *and* $M(B)$ *are not isomorphic.*

This example is best understood by thinking of the surfaces S_r for small r as being obtained by letting the length of the generator of the homology group tend to zero. To make this precise we need a notion of the length of a homology class.

III. **Lengths of cycles.** The appropriate notion of length in this context is that of "harmonic length" introduced by Landau and Osserman [3].

Let S be a Riemann surface and γ a smooth closed curve on S. The harmonic length of γ, $l(\gamma)$ is defined by $l(\gamma) = \sup\ \{\int_\gamma *du;\ u$ a real harmonic function on S, $\sup_S|u| \leq 1\}$. ($*du$ is the harmonic conjugate of the differential du.) $l(\gamma)$ is seen to depend only on the homology class of γ. One easily checks that if γ_r is a generator of the integer homology group of the surfaces S_r in the previous example then $l(\gamma_r) \longrightarrow 0$ as $r \longrightarrow 0$.

Let $S_1, S_2, \cdots$ be a sequence of homeomorphic Riemann surfaces. Let

$$h(\{S_n\}) = \lim_{\epsilon\to 0}\ \overline{\lim_{n\to\infty}}\ h_\epsilon(S_n)$$

where $h_\epsilon(S_n)$ is the dimension of the span in $H_1(S_n, R)$ of the set of γ in $H_1(S_n, \mathrm{Z})$ with $l(\gamma) < \epsilon$. Thus $h(\{S_n\})$ measures the number of linearly independent homology classes which are being pinched as n becomes large.

IV. **The main result.** By a degenerate Riemann surface we mean the object obtained from a possibly disconnected Riemann surface by making finitely many identifications of finite point sets. Let S_1 be a Riemann surface of genus g which has k boundary contours.

THEOREM 2. *Let* $S_1, S_2, \cdots$ *be a sequence of Riemann surfaces homeomorphic to* S_1 *such that* $\{S_n\}$ *is a Cauchy sequence in* $(R(S_1), d)$.

There is a Banach space A_∞ *such that* $d(A_\infty, A(S_n)) \longrightarrow 0$ *as* $n \longrightarrow \infty$. A_∞ *has the following additional properties.* A_∞ *is a point separating algebra of continuous boundary value analytic functions on the connected degenerate Riemann surface* S_∞ (= *maximal ideal space of* A_∞). ∂S_∞, *the boundary of* S_∞, *consists of* k *circles. The closure of the space of real parts of functions in* A_∞ *in the space of real valued continuous functions on* ∂S_∞ *has codimension* $2g + k - 1$. *The dimension of the homology group* $H_1(S_\infty, \mathbf{R})$ *is* $2g + k - 1 - h(\{S_n\})$.

The proof uses results on almost isometries of function algebras [6] to construct the algebra A_∞ and then uses techniques from the theory of function algebras to describe the maximal ideal space of A_∞.

One consequence of this result is that if no cycles are pinched to zero then the surfaces are not degenerating.

COROLLARY. *If* $h(\{S_n\}) = 0$ *then* S_∞ *is a Riemann surface homeomorphic to* S_1.

V. **Another example.** Let S be a compact Riemann surface of genus 1. Let p be a point of S. One can choose a local uniformizer w at p so that the surfaces $S_n = \{q \in S \mid w(q) \geqslant 1/n\}, n = 1, 2, \cdots$, satisfy the hypotheses of Theorem 2. In this case the algebra A_∞ will be the algebra of all functions f which are continuous on $\{|z| \leqslant 1\}$, analytic in $\{|z| < 1\}$, and satisfy $f'(0) = 0$.

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