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# ANALYSIS AND SYNTHESIS FOR THE POINT AND UNITARY SPECTRA. II

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This is a direct sequel to Part I [1], continuing presentation of results from a portion of [2]. The terminology and notation of [1] are retained. Further, we will denote by $E_\lambda$ the subspace $\operatorname{Ker}(T - \lambda I)$ of eigenvectors of the operator $T$ belonging to $\lambda$; by $R(\lambda, T)$, the resolvent $(\lambda I - T)^{-1}$ of $T$. The notation for "closed linear span of $\cdots$" will be "$\mathcal{L}(\cdots)$".

This section gives solutions of the problem of analysis-synthesis for complete operators whose point spectrum $\sigma_p(T)$ is all or mostly in the circle $\Gamma$ and which are close to being isometric. We will call an operator '$\Gamma$-complete' in case $\mathcal{L}(E_\lambda: \lambda \in \Gamma) = X$. Various conditions of closeness to isometricity are considered: 'strong ergodicity' ($K(T) = \sup_{n \geqslant 0} \|T^n\|_X < +\infty$); '$C$-ergodicity'

$$\left(\sup_{\lambda \in \Gamma, n \geqslant 0} \left\| \frac{1}{n+1} \sum_{k=0}^{n} T^k \lambda^k \right\|_X < +\infty \right);$$

'$A$-ergodicity' ($\sup_{|\lambda|>1} \|R(\lambda, T\|_X(|\lambda| - 1) < +\infty$); and others. Any of these conditions permits application of an ergodic theorem to ensure existence of a "spectral projector" $P_\lambda = \lim_{r \to 1+0} (r-1) R(r\lambda, T)$ onto the eigenspace $E_\lambda$ along all the other such subspaces: $P_\lambda P_\mu = 0$ if $\lambda \neq \mu$. This makes it possible to associate to every vector $x \in X$ its formal Fourier series

$$(*) \qquad x \sim \sum_{\lambda \in \sigma_p(T)} P_\lambda x.$$

The study of $\Gamma$-complete operators leads to a far-reaching analogy with the study of the group of rotations of $\Gamma$ acting in the spaces of classical analysis ($L^p$, $l^p$, Orlicz spaces, etc.). It lies near at hand to supplement the definitions given in [1]: setting $E(x) = \mathcal{L}(T^n x: n \geqslant 0)$, we will say that the

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$A$-ergodic operator $T$ 'admits harmonic analysis' in case $P_\lambda x \in E(x)$ for all $\lambda \in \Gamma$; that it 'admits harmonic synthesis' in case $E(x) = L(P_\lambda x\colon \lambda \in \Gamma)$.

The role of Fourier analysis and almost-periodicity in spectral theory of this sort was elucidated by E. R. Lorch, K. de Leeuw and I. Glicksberg, and Ju. I. Ljubič; see [3] and [4].

THEOREM 3. *Let* $X$ *be a reflexive space*; $\gamma$, *a subgroup of* $\Gamma$; *and* $T$, *a strongly ergodic operator on* $X$. *Then there is a projector* $P_\gamma$ *of* $X$ *onto* $E_\gamma = L(E_\lambda : \lambda \in \gamma)$, *which commutes with every operator commuting with* $T$; *furthermore,* $\|P_\gamma\| \leqslant K(T)$.

The proof of this theorem provides also a proof that such $T$ admit harmonic synthesis. Another sequence is the decomposability of the part of the operator belonging to $\sigma_p(T) \cap \Gamma$. It is shown by examples that the hypothesis of reflexivity as well as the arithmetic nature of the set $\gamma$ are essential.

The possibility of harmonic analysis is evident for any $A$-ergodic operator. Distinctly less trivial is the problem of harmonic synthesis; it is solved in the following theorems, in which ergodicity requirements occur in "two-sided" forms, that is, bear on $R(\lambda, T)$ for both $|\lambda| > 1$ and $|\lambda| < 1$, or on $\|T^n\|$ for all integral $n$.

THEOREM 4. *In the reflexive space* $X$, *consider an A-ergodic operator* $T$ *with A-ergodic inverse, i.e.* $\sup_{|\lambda| \neq 1} \operatorname{dist}(\lambda, \Gamma)\|R(\lambda, T)\| < +\infty$. *If the spectrum of* $T$ *is at most countable then* $T$ *is* $\Gamma$*-complete. In particular, such an operator admits spectral and harmonic synthesis.*

THEOREM 5. *Let* $X$ *be a Hilbert space*, $\dim X = +\infty$. *Let* $\alpha_n > 1$ *with* $\lim_{n\to\infty} \alpha_n = +\infty$, *and let* $\mu$ *be a positive decreasing function on* $(0, +\infty)$ *with* $\mu \geqslant 1$ *and* $\lim_{x\to 0} \mu(x) = +\infty$. *Then there exists an operator* $T$ *on* $X$ *such that* (a) $X = L(E_\lambda\colon \lambda \in \Gamma)$; (b) $\sigma(T)$ *is countable*; (c) $\|T^n\| \leqslant \alpha_{|n|}$ $(n = 0, \pm 1, \cdots)$; (d) $\|R(\lambda, T)\| \leqslant \mu(\operatorname{dist}(\lambda, \Gamma))/\operatorname{dist}(\lambda, \Gamma)$ *for all* $\lambda \notin \Gamma$; (e) $T$ *admits spectral analysis*; (f) $T$ *does not admit spectral synthesis.*

THEOREM 6. *Let* $T$ *be a linear operator on a Banach space which may be extended to a complete operator with spectral radius less than* 1. *Then there exists also a C-ergodic* $\Gamma$*-complete extension of* $T$.

Recall that by Theorem 1 [1] $T$ might be in particular any compact operator with $\|T\| < 1$. Theorem 6 shows that $C$-ergodicity and $\Gamma$-completeness are consistent with failure of harmonic synthesis and spectral analysis.

Returning to results in the "positive" direction—summability of $(*)$ by any linear method yields the possibility of harmonic (and, a fortiori, spectral) synthesis. In this section of [2] there are analogues of classical theorems of H. Lebesgue, Dini-Lipschitz, O. Szász, and S. N. Bernšteĭn on convergence (and on summability by various methods) of Fourier series in eigenfunctions of strongly ergodic operators.

We also investigate the connection between the unitary point spectrum $\sigma_p(T) \cap \Gamma$ and other properties of the operator. The first problem here is 'localization of the point spectrum', by which we mean finding all $\Gamma$-complete operators $T$ such that $\sigma(T)$ is just the closure of $\sigma_p(T) \cap \Gamma$.

THEOREM 7. (I) *Let $T$ be an invertible linear operator on the Banach space $X$; assume $X = L(E_\lambda : \lambda \in \Gamma)$ and*

$$\sum_{n \in Z} (n^2 + 1)^{-1} |\log \|T^n\|| < +\infty;$$

*then* $\sigma(T) = \overline{\sigma_p(T) \cap \Gamma}$.

(II) *Let $X$ be a Hilbert space*, $\dim X = +\infty$. *Let $\{w_n\}_{n \in \mathbf{Z}}$ be a sequence with the properties that $1 \leqslant w_{n+k} \leqslant w_n w_k$ $(n, k \in \mathbf{Z})$, that $\log w_n = o(n)$, that $n^p = o(w_n)$ for any $p > 0$, and that $\Sigma_{n \in \mathbf{Z}} (n^2 + 1)^{-1} \log w_n = +\infty$. Then there exists an operator $T$ on $X$, bounded and boundedly invertible, such that* (a) $\|T^n\| \leqslant w_n$ $(n \in \mathbf{Z})$; (b) $X = L(E_\lambda : \lambda \in \Gamma)$; (c) $\sigma(T) = \Gamma$; (d) $\sigma_p(T)$ *is a countable nowhere dense subset of* $\Gamma$.

THEOREM 8. (I) *Let $T$ be a $\Gamma$-complete strongly ergodic operator on the Banach space $X$. Then* $\overline{\sigma_p(T) \cap \Gamma} = \sigma(T)$. (II) *Let $X$ be a Hilbert space*, $\dim X = +\infty$. *Assume that $w_n \nearrow \infty$ and that the sequence $\{\log w_n\}$ is positive and concave. Then there exists a C-ergodic operator $T$ on $X$ such that* (a) $\|T^n\| \leqslant w_n$ $(n \geqslant 0)$; (b) $L(E_\lambda : \lambda \in \Gamma) = X$; (c) *$T$ admits harmonic synthesis*; (d) $0 \in \sigma(T)$.

Thus localization of the spectrum by "two-sided" conditions is related to the known nonquasianalyticity criterion $\Sigma_{n \in \mathbf{Z}} (n^2 + 1)^{-1} |\log \|T^n\|| < +\infty$; if one uses "one-sided" conditions instead, this criterion is replaced by strong ergodicity.

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