BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 1, January 1975

ON THE CONJUGATE OF BOUNDED FUNCTIONS[1]

BY S. K. PICHORIDES

Communicated July 2, 1974

ABSTRACT. If f is a real 2π-periodic function such that $|f| \leqslant k < \pi/2$ and $\tilde{f}$ its conjugate, then $\|\sinh(\tilde{f}/2)\|_2 \leqslant (\cos k)^{-1/2}\|f/2\|_2$.

Let f be a real 2π-periodic function and $\tilde{f}$ its conjugate. If $|f| \leqslant 1$ and $0 < k < \pi/2$, then [3, VII, 2.11]

$$\frac{1}{2\pi}\int \exp(k|\tilde{f}|) \leqslant \frac{2}{\cos k}. \tag{1}$$

It follows from (1) that if $m(y) = |\{x: |\tilde{f}(x)| > y\}|$ is the distribution function of $\tilde{f}$, then

$$m(y) \leqslant \mathrm{Const}/\exp(ky). \tag{2}$$

In the case where f is the characteristic function of a measurable set $E \subset [-\pi, \pi]$, E. Stein and G. Weiss proved that [2, Lemma 5]

$$m(y) \leqslant \mathrm{Const}\,\sin(|E|/2)/\sinh(y/2). \tag{3}$$

Moreover they gave an exact formula for $m(y)$, which shows that $m(y)$ depends on $|E|$ only.

In [3] the proof of (1) is based on Cauchy's theorem for holomorphic functions. The same method can provide an alternative proof of the theorem of Stein and Weiss mentioned above [1, III, 1.10]. It seems that it has passed unnoticed that this method also yields an inequality similar to (3) for all bounded functions. It is obvious that such an inequality considerably improves (2) for large values of y. A convenient way to formulate this result is the following

THEOREM. *If f is a real 2π-periodic function such that $|f| \leqslant k < \pi/2$ and $\tilde{f}$ its conjugate, then*

AMS (MOS) subject classifications (1970). Primary 42A40, 44A15.

[1]This work was done in part while the author was a visitor at the Department of Mathematics, Université de Paris–Sud, 91405 Orsay, France.

$$\|\sinh(\tilde{f}/2)\|_2 \leqslant (\cos k)^{-1/2} \|f/2\|_2. \tag{4}$$

PROOF. Let $u + iv$ be the holomorphic function in the open unit disc associated with f. The real part $(\cosh v)\cos u$ of $\cos(u + iv)$ is harmonic. Observing that $v(0) = 0$ and integrating over $|z| = r < 1$ we have

$$\frac{1}{2\pi}\int(\cosh v)\cos u = \cos u(0) = \cos\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx\right].$$

Using the obvious inequality $\cos u \geqslant \cos k$, we obtain

$$\begin{aligned}\cos k\|\sinh(v/2)\|_2^2 &\leqslant \frac{1}{4\pi}\int(\cosh v - 1)\cos u \\ &= \left[\cos u(0) - \frac{1}{2\pi}\int\cos u\right]\Big/2 \\ &= \frac{1}{2\pi}\int\sin^2(u/2) - \sin^2[u(0)/2].\end{aligned}$$

Letting $r \longrightarrow 1$ we obtain

$$\|\sinh(\tilde{f}/2)\|_2^2 \leqslant \frac{1}{\cos k}\left\{\frac{1}{2\pi}\int\sin^2\frac{f}{2} - \sin^2\left[\frac{1}{2\pi}\int\frac{f}{2}\right]\right\}, \tag{5}$$

from which (4) follows immediately. □

REMARK. It is obvious that for any function f with constant absolute value k (i.e. $f = k(2g - 1)$, where g is the characteristic function of a measurable set), (5) reduces to equality.

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Current address: Université de Paris-Sud, Centre d'Orsay, 91405 Orsay, Mathématique, Bâtiment 425, France