ON THE DISTRIBUTION OF THE ZEROS OF
THE RIEMANN ZETA FUNCTION IN SHORT INTERVALS

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Here we are concerned with the distribution of the zeros of the Riemann zeta function $\zeta(s)$ in short intervals in the vertical direction. We establish a mean value theorem for the number of the zeros in short intervals and derive several consequences from it, in particular, about the difference between the ordinates of the zeros and properties of uniform distribution of the zeros. Details will appear later.

We start from the Riemann-von Mangoldt formula for the number $N(t)$ of the zeros of $\zeta(s)$ in $0 < \mathrm{Im}\, s \leqslant t, 0 < \mathrm{Re}\, s < 1$, where the possible zeros on $\mathrm{Im}\, s = t$ are counted with weight one half:

$$N(t) = L(t) + S(t) \quad \text{for} \quad t > t_0,$$

where

$$L(t) = \frac{t}{2\pi} \log t - \frac{1 + \log 2\pi}{2\pi} t + \frac{7}{8} + O\left(\frac{1}{t}\right),$$

Here

$$S(t) = (1/\pi) \arg \zeta(\tfrac{1}{2} + it),$$

where $\arg \zeta(\tfrac{1}{2} + it)$ is defined by continuous variation on the half line $\sigma + it, \sigma \geqslant \tfrac{1}{2}$ starting with the value zero if t is not the ordinate of a zero of $\zeta(s)$. If the path crosses a zero, we put

$$\arg \zeta(\tfrac{1}{2} + it) = \tfrac{1}{2}\{\arg \zeta(\tfrac{1}{2} + i(t + 0)) + \arg \zeta(\tfrac{1}{2} + i(t - 0))\}$$

(cf. [5, 9.2.]). Then the number

$$\Delta_h N(t) \equiv N(t + h) - N(t) = L(t + h) - L(t) + S(t + h) - S(t)$$

of the zeros of $\zeta(s)$ in a short interval $t < \mathrm{Im}\, s \leqslant t + h$ essentially depends

on $S(t + h) - S(t)$. We can show, by modifying A. Selberg's approach (cf. [3]):

LEMMA. *For* $T > 1$ *and* h *in* $0 < h \leqslant T - T^{1/16}$, *we have*

$$\int_0^T (S(t + h) - S(t))^l \, dt = \frac{2k!}{(2\pi)^{2k} k!} \, T(2 \log (3 + h \log T))^k$$

$$+ \, O((Ak)^{4k} T \cdot (\log (3 + h \log T))^{k - \frac{1}{2}}) \text{ for } l = 2k,$$

$$= O((Ak)^{3k} T(\log (3 + h \log T))^{k - 1}) \text{ for } l = 2k - 1.$$

Hence we get a mean value theorem for $N(t + h) - N(t)$ *also.*

An immediate consequence of this is a result about the normal density of the zeros in short intervals.

THEOREM 1. *If* $h \log T$ *tends to* ∞ *as* T *tends to* ∞, *then*

$$\frac{(h/2\pi) \log T}{\Phi(T)} < N(t + h) - N(t) < \Phi(T)\frac{h}{2\pi} \log T$$

for any positive increasing function $\Phi(T)$ *which tends to* ∞ *as* T *tends to* ∞ *and for almost all (in the sense of the density)* t *in* $0 < t \leqslant T$.

For shorter intervals we can show

THEOREM 2.

$$N\left(t + \frac{2\pi C}{\log T}\right) - N(t) > C + C_1(\log C)^{\frac{1}{2}}(\log \log C)^{\frac{1}{2} + \epsilon}$$

for a positive proportion of t *in* $(0, T)$, *and*

$$N\left(t + \frac{2\pi C}{\log T}\right) - N(t) < C - C_1(\log C)^{\frac{1}{2}}(\log \log C)^{\frac{1}{2} + \epsilon}$$

for a positive proportion of t *in* $(0, T)$, *where* $\epsilon > 0, C > C_0, C_0$ *and* C_1 *are suitable positive absolute constants.*

If we denote the ordinate of the nth zero of $\zeta(s)$ by γ_n, where $0 \leqslant \gamma_n \leqslant \gamma_{n+1}$, the following is a consequence of Theorem 2.

COROLLARY. *For each* $r = 1, 2, 3, \cdots$ *a positive proportion of* γ_n's *satisfy*

$$\frac{\gamma_{n+r} - \gamma_n}{r} \, \frac{\log \gamma_n}{2\pi} < 1 - A,$$

and also a positive proportion of γ_n's *satisfy*

$$\frac{\gamma_{n+r} - \gamma_n}{r} \; \frac{\log \gamma_n}{2\pi} > 1 + A,$$

where A is a positive absolute constant less than 1 which may depend on r. In fact we can take $A = \exp(-C_2 r^2)$ with a positive absolute constant C_2.

Also from our Lemma we see

THEOREM 3.

$$\sum_{0 < \gamma_n \leqslant T; \gamma_{n+1} - \gamma_n > h} (\gamma_{n+1} - \gamma_n) \leqslant A \frac{T \log(3 + h \log T)}{(h \log T)^2}$$

uniformly for a positive h, and

$$\sum_{0 < \gamma_n \leqslant T} (\gamma_{n+1} - \gamma_n)^2 \leqslant A \frac{T}{\log T},$$

where A's are positive absolute constants.

About the property of uniform distribution of the γ_n's we can show

THEOREM 4. *If* $h \log T$ *tends to* ∞ *as* T *tends to* ∞, *then*

$$\sum_{m=0}^{[T/h]} (\Delta_{\alpha h} - \alpha \Delta_h) N(mh) = o(N(T))$$

for each α in $0 < \alpha \leqslant 1$.

In particular, if $h = 1$, this implies $\{\gamma_n : n = 1, 2, \cdots\}$ is uniformly distributed mod 1.

From a mean value theorem of $\Delta_h N(t)$, we can get some information about the multiplicities of zeros. Namely, we can show

THEOREM 5. *For any* j *in* $1 \leqslant j \leqslant \sqrt{\log T}$, *we have*

$$\sum_{j=1}^{\infty} j^{2k} N_j(T) \leqslant CN(T) \quad \text{for} \quad T > T_0,$$

where $N_j(T)$ is the number (counted without multiplicity) of zeros of $\zeta(s)$ in $0 < \operatorname{Im} s \leqslant T, 0 < \operatorname{Re} s < 1$ with the multiplicity exactly j.

From this we see

COROLLARY.

$$\varlimsup_{T \to \infty} \frac{1}{N(T)} \sum_{j=K}^{\infty} jN_j(T) \leqslant e^{-A\sqrt{K}}$$

for $K > K_0$, A and K_0 are suitable positive absolute constants.

Finally we may remark here that the results in this paper have been extended to Dirichlet L-functions and also to Dirichlet series attached to cusp forms under certain hypotheses. q-analogues of this paper have been also obtained. These will also appear later.

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