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BOOK REVIEW

Invariant subspaces, by H. Radjavi and P. Rosenthal, Springer-Verlag, New York, 1973, xi+219 pp., DM 50.—

In recent years, a large number of invariant subspace questions have been raised, and some of them answered, in an effort to make some progress towards structure theorems for operators and algebras of operators. Typically, a collection $\mathscr{A}$ of bounded linear operators on a Hilbert space $\mathscr{H}$ is specified. The lattice of $\mathscr{A}$ is to the collection of all closed linear subspaces M of $\mathscr{H}$ which are invariant under $\mathscr{A}$; that is, $A(M)\subset M$ for each A in $\mathscr{A}$. What is the lattice of $\mathscr{A}$ like? In particular, must the lattice contain more than $\{0\}$ and $\mathscr{H}$ if $\mathscr{A}$ consists of a single operator (the invariant subspace question), or if A is the algebra of operators which commute with a particular operator (the hyperinvariant subspace question), or if A is any algebra of operators (the transitive algebra question)? The book under review is a report on the progress towards some adequate answers to these questions.

The book is carefully organized and sufficiently self-contained to be accessible to most mathematicians. Background material which is not part of a standard graduate course and which is not explicitly developed is at least discussed and referred to. There are no long proofs whose punch lines come out of nowhere. Some results which are pertinent to the subject of this book, but which are parts of elaborate theories, have been omitted. For these results, the reader is referred to specific accounts of these theories by, for example, Sz.-Nagy-Foiaş or Foiaş-Colojoară. (See Bull. Amer. Math. Soc. 77 (1972), 938–942.) At the end of each chapter is an Additional Propositions section of homework problems for the reader, and a Notes and Remarks section in which credit for theorems is given and additional results are mentioned.

The book begins with the usual Chapter 0 in which terminology and notation are set and background material is discussed. The first few real chapters cover standard topics in operator theory, much of which appears elsewhere in book form. Chapter 1 on normal operators gives a nice proof of the spectral theorem which hints at Banach algebras without mentioning them. This chapter also includes an easy proof of Fuglede's theorem based on Rosenblum's study of the operator equations $AX-XB=Y$. Chapter 2 develops the Riesz functional calculus and gives a useful comparison due to Crimmins and Rosenthal of the lattice of an operator and the lattice of an analytic function of that operator.

One of the oldest and most developed parts of invariant subspace theory is the study of shift operators. From an analytic point of view, the shift operator of multiplicity α is multiplication by z on the Hilbert space of H^2 functions on the disc taking values in a Hilbert space of dimension α. In Chapter 3 of this book, it is shown that the operators which commute with a shift operator are just operators of "multiplication" by bounded analytic operator valued functions, while the invariant subspaces for the shift are the ranges of these operators. There is also some finer analysis of the invariant subspaces of the shift of multiplicity one. The last section of this chapter contains the deBrange-Rovnyak-Foiaş theorem that every contraction operator is the restriction of the adjoint of a shift operator to an invariant subspace. This theorem reduces the general invariant subspace question to a finer study of the invariant subspaces of shift operators, and thus to a factorization problem for operator valued H^∞ functions.

In the next chapter, the basic question is given a new twist: Given a lattice $\mathscr{L}$, is there an operator whose lattice is $\mathscr{L}$? This is really two questions, one in which $\mathscr{L}$ is an abstract lattice and "is" means "is order isomorphic to", and another in which $\mathscr{L}$ is a lattice of subspaces of a Hilbert space and "is" means "is". Included in this chapter are the Donoghue shift whose lattice is $\omega+1$ and the Volterra operator whose lattice is $[0, 1]$.

The next two chapters are devoted to showing that particular kinds of operators have invariant subspaces. Chapter 5 treats compact and other quasitriangular operators. The deepest results in Chapter 6 pertain to operators of the form $A+B$ where A is normal and B is in one of the Schatten classes.

In the final three chapters minus one, the emphasis shifts from operators to algebras of operators. In particular, to what degree is an algebra of operators (weakly closed and containing the identity operator) characterized by its invariant subspaces? It is not the case that any two such algebras with the same lattice are the same. Then when is an algebra the largest algebra with its particular lattice? (Which algebras are reflexive?) When does a lattice belong to only one algebra? In particular, is the algebra of all operators the only weakly closed algebra whose lattice is $\{\{0\}, \mathscr{H}\}$ (the transitive algebra problem)? Are selfadjoint algebras the only algebras whose lattices are closed under orthocomplementation (the reductive algebra problem)?

For the purposes of this book, Chapter 7 is a technical chapter. It includes a little of the theory of von Neumann algebras with applications to n-normal operators. Chapters 8 and 9 apply a formula developed by

Arveson to the problems mentioned above. Arveson observed that an operator B is in an algebra $\mathscr{A}$ if for every positive integer n, the lattice of $\mathscr{A}^{(n)}$ is included in the lattice of $B^{(n)}$, where $B^{(n)}$ is the direct sum of n copies of B and $\mathscr{A}^{(n)}$ is the algebra of all $A^{(n)}$ where A is in $\mathscr{A}$. Furthermore, in most cases it suffices to consider subspaces of the form

$$\{(x, T_1x, T_2x, \cdots, T_{n-1}x) : x \in \mathscr{D}\},$$

where $\mathscr{D}$ is a linear manifold in $\mathscr{H}$ and the T_i are linear transformations which commute with $\mathscr{A}$. This program yields theorems like: The algebra of all operators is the only transitive algebra which contains a maximal abelian selfadjoint subalgebra, or a shift of finite multiplicity, or a finite rank operator, or a Donoghue shift. The major result in these chapters whose proof does not follow this format is due to Lomonosov. It says that there is only one transitive algebra which contains a compact operator. This is a consequence of the Schauder fixed point theorem.

The very last chapter is a discussion of some open questions.

Invariant subspaces is a carefully written, nicely organized book. Theorems are clearly and accurately stated. Proofs are logical and understandable with highlights properly emphasized. Lemmas are well chosen. Many of the proofs have been substantially changed from the originals in order to unify the development. There are surprisingly few typographical errors. The writing style is curt and businesslike with a minimum of conversation between theorems. This theorem, proof, theorem, proof style makes casual reading difficult, but it is not inappropriate for a book of this type.

The only real weakness of this book is its publication date. As a progress report, it can expect an eventual partial obsolescence. Indeed, its obsolescence would mark its success. But it is unfortunate that the publication of this book matches so closely some important developments in this area. The quasitriangular operators mentioned in Chapter 5 were originally singled out in order to get a little more out of the von Neumann-Aronszajn-Smith proof that compact operators have nontrivial invariant subspaces. But the consequences of Lomonosov's lemma in Chapter 8 are much stronger than anything in Chapter 5 and do not depend on quasitriangularity. So it appears that we do not need to know about quasitriangularity. But Apostol, Foiaş, and Voiculescu have recently proved that the adjoint of any nonquasitriangular operator has point spectrum and therefore every nonquasitriangular operator has invariant, even hyperinvariant, subspaces. So now it is exactly quasitriangular operators that we have to know about. This interesting turn of events is hidden in this book. The Apostol, Foiaş, Voiculescu theorem is mentioned in a Notes and Remarks section. It is both a recent and complicated result, so mention is probably all that was possible. Lomonosov's theorem is even more recent, but it is comparatively simple and so the authors managed to include it in Chapter 8, separated from quasitriangularity by two chapters. The authors have probably included as much of this as was possible, but it is too bad they have hidden rather than emphasized the interrelationships.

Another interesting result which timing has relegated to the Notes and Remarks section is Davie's work on the Bishop operator. The Bishop operator on $L[0, 1]$, defined by $B(f)(x)=xf(x+a)$, where a is fixed and $+$ is addition modulo one, has long been a leading candidate for an operator without proper invariant subspaces. Davie has shown that, for most a, B has a proper invariant subspace.

Radjavi and Rosenthal have done an admirable job of gathering together, rewriting and unifying much of the literature related to the invariant subspace question. Their account is readable and, except as mentioned above, complete. Anyone who is interested in learning about this problem would profit from this book.

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