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THE HOPF RING FOR COMPLEX COBORDISM[1]

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It is our purpose here to announce the results of our study of the homology of the spaces in the Ω-spectrum for complex cobordism and Brown-Peterson cohomology. Let $MU(n)$ be the standard Thom complex. $\mathrm{M}U_k = \lim_{n\to\infty} \Omega^{2(n-k)}MU(n)$ is the $2k$ space in the Ω-spectrum for complex cobordism. We will consider the space $\mathrm{M}U = \lim_{n\to -\infty} \Pi_{j>n} \mathrm{M}U_j$. We find this product easier to study than the separate factors, as will become apparent below.

For a space X we have $[X, \mathrm{M}U] \simeq U^{2*}(X)$, the even degree part of the complex cobordism of X. Because MU is a multiplicative theory, $U^{2*}(X)$ is a ring and $\mathrm{M}U$ is a commutative ring with identity in the homotopy category. Thus we have that for any field k, $H_*(\mathrm{M}U; k)$ is a commutative ring with identity in the category of k-coalgebras, i.e., it is a "Hopf ring".

In more common language, the homology has two products and a coproduct. $\circ$ will denote the multiplicative product which comes from the ring structure on the spectrum, while $*$ will denote the additive product coming from the loop structure $(\Omega^2\mathrm{M}U \simeq \mathrm{M}U)$. They obey the following distributive law: if $\psi(z) = \Sigma\, z' \otimes z''$ is the coproduct, then $z \circ (x * y) = \Sigma\, (z' \circ x) * (z'' \circ y)$.

We now describe the structure of $H_*(\mathrm{M}U; R)$ where R is an algebra over a field k. Let

$$C_R(X) = \left\{ x \in \prod_{i\geq 0} H_i(X; R) \colon \psi(x) = x \mathbin{\hat{\otimes}} x,\, x \neq 0 \right\}.$$

$C_R(\mathrm{M}U)$ is a ring, and for each $x \in C_R(X)$ we have a ring homomorphism $\lambda_x\colon U^{2*}(X) \longrightarrow C_R(\mathrm{M}U)$ defined by $\lambda_x(u) = u_*(x)$ for $u \in U^{2*}(X)$. Let

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$\beta_n \in H_{2n}(CP^\infty; R)$ be the standard generator. $\beta(r) = \Sigma \beta_i r^i \in C_R(CP^\infty)$ for $r \in R$.

$U^{2*}CP^\infty = U^*[[T]]$, the power series ring on the canonical generator $T \in U^2 CP^\infty$ over the coefficient ring $U^* = Z[x_2, x_4, \cdots]$, a polynomial algebra on negative even-dimensional generators. We now have

$$b(r) = \sum b_i r^i = \lambda_{\beta(r)}(T) \in C_R(MU).$$

(In other words, if we represent T by a map $f\colon CP^\infty \longrightarrow MU$, $f_*(\beta_n) = b_n$.)

Note that $\pi_0 MU \simeq \pi_* MU \simeq U_* \simeq U^{-*}$, and any element $a \in U^*$ or U_* gives rise to an element $[a] \in H_0 MU$. The $[x_{2i}]$ generate the Hopf ring $H_0 MU$. Under the standard multiplication $CP^\infty \times CP^\infty \longrightarrow CP^\infty$, T pulls back to $\Sigma a_{ij} T_1^i \otimes T_2^j$, where $a_{ij} \in U^{2(1-i-j)}$. The a_{ij} are the coefficients of the formal group associated with complex cobordism (see [1]).

We use the above multiplication to get our first theorem.

THEOREM 1. *In* $C_R(MU)$,

$$b(r_1 + r_2) = \sum_{i,j>0} [a_{ij}]\, b(r_1)^i b(r_2)^j.$$

The following is just a restatement of the theorem.

THEOREM 1′. *In* $H_*(MU; R)$,

$$b(r_1 + r_2) = \mathop{*}_{i,j>0} ([a_{ij}] \circ b(r_1)^{\circ i} \circ b(r_2)^{\circ j}).$$

COROLLARY 2. $\log b(r_1 + r_2) = \log b(r_1) + \log b(r_2)$ *in* $C_R(MU) \otimes Q$, *where*

$$\log X = \sum_{n>0} \frac{[CP^{n-1}]}{n} X^n.$$

If we are working over the integers we can rephrase this to:

COROLLARY 2′. $\log b(r) = b_1 r$ *in* $QH_*(MU; Q[r])$.

Let $H_R\, MU$ denote the Hopf ring generated by the $[x_{2i}]$ and the b_n subject to the relations implied by Theorem 1.

THEOREM 3. *The map* $H_R\, MU \longrightarrow H_*(MU; R)$ *is a Hopf ring isomorphism.*

This is still true if we replace R by Z.

The main result of [6], where the investigation of the homology of $\mathbf{M}U_k$ was begun, is now an immediate corollary of Theorem 3.

COROLLARY 4. *$H_*(\mathbf{M}U_k; Z)$ has no torsion.*

PROOF. $H_R\,\mathbf{M}U$ has only even-dimensional elements.

Theorem 3 is a total information result. Not only does it give a complete description of both products and the coproduct, but, using the results of Switzer [5] on the coaction of the dual of the Steenrod algebra on $\mathbf{C}P^\infty$, we can compute the structure of $H_*(\mathbf{M}U; F_p)$ as a comodule over the dual to the Steenrod algebra directly from our algebraic construction $H_R\mathbf{M}U$.

The most difficult part of the proof of Theorem 3 is showing that the map is onto. To do this, we first replace MU by BP, the Brown-Peterson spectrum [2], [3]. We can recover information about MU from BP by Quillen's result that $U^*(X)_{(p)} \simeq U^*_{(p)} \otimes_{BP^*} BP^*(X)$. There are, of course, analogues of Theorems 1–4 for the analogous space $\mathbf{B}P$. We have

$$H_*(\mathbf{M}U; F_p) \simeq H_0(\mathbf{M}U; F_p) \otimes_{H_0(\mathbf{B}P;F_p)} H_*(\mathbf{B}P; F_p)$$

and $BP_* \simeq \pi_*BP \simeq Z_{(p)}[v_1, v_2, \cdots]$, where v_s is a $2(p^s - 1)$-dimensional generator. From now on, all homology groups will have coefficients in F_p. An immediate consequence of Theorem 1 is that all the b_i can be expressed in terms of b_{p^n}, which we denote by $b_{(n)}$. Note that these elements generate the stable homology H_*BP. Define

$$v^Ib^J = [v_1^{i_1}v_2^{i_2} \cdots] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \circ \cdots,$$

where $I = (i_1, i_2, \cdots)$ and $J = (j_0, j_1, \cdots)$ are sequences of nonnegative integers, and $b_{(n)}^{\circ j_n}$ denotes the j_nth power of $b_{(n)}$ under the multiplicative or $\circ$ product.

DEFINITION. v^Ib^J is called allowable if

$$J = p\Delta_{k_1} + p^2\Delta_{k_2} + \cdots + p^n\Delta_{k_n} + J' \text{ (nonneg. seq.)}, \qquad k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$$

implies $i_n = 0$. (Δ_k is the sequence with 1 in the kth place and zeros elsewhere.)

Let $\mathbf{B}P_{(0)}$ denote the zero component of $\mathbf{B}P$. $H_*\mathbf{B}P_{(0)}$ is a Hopf algebra under the $*$ product. Let Q and P denote the indecomposables and primitives respectively. We now have

THEOREM 5. (a) $H_*\mathbf{B}P_{(0)}$ *is a polynomial algebra.*

(b) *The allowable* $v^I b^J$ $(J \neq 0)$ *form a basis for* $QH_*\mathbf{B}P_{(0)}$.

(c) *The* $v^I b^{J+\Delta_0}$ with $v^I b^J$ *allowable* (J *possibly zero*) *form a basis for* $PH_*\mathbf{B}P_{(0)}$.

The proof of Theorem 5 is by induction on dimension, using Eilenberg-Moore spectral sequences which go from $H_*\mathbf{B}P_{(0)}$ to $H_*\Omega\mathbf{B}P_{(0)}$ and back to $H_*\mathbf{B}P_{(0)}$ using the periodicity $\Omega^2\mathbf{B}P_{(0)} \simeq \mathbf{B}P$ and Theorem 6.

$H_*\mathbf{B}P$ is a BP_* module under the $\circ$ product as $BP_* \subset H_0(\mathbf{B}P)$. We have the ideal $(v_1, v_2, \cdots) = I \subset BP_*$. The $[p]$-sequence $[p](X)$ can be defined by $\log_{BP} [p](X) = p \log_{BP}(X)$. Also, $[p](T)$ is the image of T when pulled back by the pth power map $\mathbf{C}P^\infty \longrightarrow (\mathbf{C}P^\infty)^p \longrightarrow \mathbf{C}P^\infty$ in $BP^*\mathbf{C}P^\infty \simeq BP^*[[T]]$. (*Note.* $b = b(1)$.)

THEOREM 6. (a) $[p](b) = 0$ *in* $C_{F_p}(\mathbf{B}P)$.

(b) $\Sigma_{i=1}^n [v_i] \circ b_{(n-i)}^{\circ p^i} = 0$ *in* $QH_*\mathbf{B}P/I^2QH_*\mathbf{B}P$.

The first statement follows from the fact that the pth power map is trivial in $H_*\mathbf{C}P^\infty$. (Recall that our coefficients are all F_p.) The second statement follows from the fact that the coefficient of X^{p^n} in the $[p]$ sequence is a $2(p^n - 1)$-dimensional generator in BP_*.

We now state some of the geometric corollaries which follow from our work.

U_*MU can be identified with the cobordism group of maps (with even codimension) of compact stably almost complex manifolds (see Stong [4] for the analogous statement in the unoriented case). From this point of view our main result is

THEOREM 7. U_*MU *is a Hopf ring generated by maps to a point, identity maps, and linear embeddings* b_n: $\mathbf{C}P^{n-1} \hookrightarrow \mathbf{C}P^n$.

COROLLARY 8. *Any map of compact stably almost complex manifolds is cobordant to one of the form* f: $\amalg_i F_i \times V_i \longrightarrow M$, *where* $f|F_i \times V_i$ *is the composition of the projection* $F_i \times V_i \longrightarrow V_i$ *and an embedding* $V_i \hookrightarrow M$.

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